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A STUDY OF CHILDREN'S LANGUAGE USE WHEN SOLVING PARTITIONING PROBLEMS: GRADES TWO THROUGH FOUR

by

©

FRANCES BERYL WALES

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SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH
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Thomas E. Kieren
External Examiner

Date July 19, 1984

DEDICATION

To the memory of my gentle mother who believing I could do anything at the price of an effort, taught me patience and persistance . . .

And the memory of my father who taught me to value education . . .

Also to my lovely Lillian.

ABSTRACT

The major intent of the study was threefold: to determine if children use oral language to help solve mathematical problems; if so, to describe the verbalizations of individual and small groups of children, grades two through four, involved in mathematical problem solving; and to identify the purpose of children's verbalizations relative to mathematical problem solving, that is, how language was used as they came to know the solutions to the problems.

Eight written partitioning problems varying in mathematical difficulty were adapted for the study.

The researcher interacted with the respondents in the context of controlled yet flexible problem solving sessions characterized by intense observation and questioning. The sessions were videotaped, audiotaped and later transcribed.

Criteria for analysis of the data were determined by the nature of the subjects' responses and in relation to the literature (theory and research findings) concerned with children's use of language and the problem solving process. Three stages of analysis occurred: ongoing (during the research session), reflective (between the research sessions) and documentive (following the data collection period).

Children, assessed by their homeroom teachers to be of at least average intelligence, average or high language users and capable of interacting, were chosen for the study. The sample, consisting of 26 children, individuals and groups of three representing each grade level, grades two through four, was drawn from an elementary-junior high school within the St. Albert Protestant Separate School District, Alberta.

From the literature, four steps in the problem solving process were identified: understanding, planning, solving and reviewing. Within each step, specific language functions and thinking strategies were evidenced in the oral language used by the children. Five main language functions emerged from the data: the children used language to clarify, report, direct, predict and self maintain. The oral language played an important role in that it allowed problem solvers to make a concerted effort to direct attention to the problem and the process as well as to clarify understandings. It seemed as though the overriding function of the language was to clarify thought.

Distinct differences in both the language-thinking strategies and mathematical-thinking strategies were apparent within the problem solving steps and in relation to the age or grade level of the problem solver. Further, levels of awareness of mathematical labels were sharply defined: grade two children knew only the terms 'one-half' and 'one quarter,' grade three children either knew or could derive one third or two thirds, but the grade four group knew all labels and were able to use interchangeably mixed numbers for improper fractions.

The interaction of the small group was lengthy, thorough and reflective. An intriguing sense of community appeared to arise, giving the group members privy to information not available to the individual problem solver or the researcher.

The significance of the study is that it provides insight into children's language use and thought while solving mathematical partitioning problems. In addition, it affords a basis for further

examination of the role of oral language used by children engaged in various problem solving situations.

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Chapter I

INTRODUCTION AND STATEMENT OF THE PROBLEM

The relationship of the role of language to mathematical problem solving has provoked concern among educators for many decades. Seventy years ago, when writing about the measurement of arithmetic ability, Thorndike (1912) stated that both mathematical insight and knowledge plus acquaintance with language were essential.

When writing about the versatility and usefulness of language Halliday (1969) states, "Language is, for the child, a rich and adaptible instrument for the realization of his intentions and there is hardly any limit to what he can do with it" (p. 27).

Hargis (1976), concerned about children with inadequate language for success in mathematics, contended that knowledge of language structure is important. In 1977, Knight and Hargis expanded the idea of the effect of language on mathematical learning. Nelson (1979) concluded that the entire area of verbalizations and their role in problem solving needs further study.

It would seem that the area of the relationship of language to mathematics is recognized as important yet remains unclear. Further study to clarify this relationship is essential.

PURPOSE OF THE STUDY

The purpose of the study is threefold: (1) to attempt to identify the purpose of children's verbalizations relative to mathematical problem solving; (2) to describe the verbalizations of children involved in mathematical problem solving, individually and in small group settings from grades two through four; and (3) to examine the role of teacher in relation to the children and the tasks involved in mathematical problem situations.

DEFINITION OF TERMS

In this study the following definitions will be adopted.

Problem solving. An all encompassing term which can mean either a process, skill or goal. For this study problem solving will be considered a process. It is the process of applying previously acquired knowledge to new and unfamiliar situations (National Council of Supervisors and Mathematics, 1977). It may also be an individually acquired set of processes brought to bear on a situation that confronts the individual (Le Blanc, 1977).

Language. All verbalizations which are expressed by the child when problem solving in mathematical situations.

High language user. A child who is rated by his language arts teacher as above average in achievement in oral language.

Average language user. A child who is rated by his language arts teachers as average in achievement in oral language.

RESEARCH QUESTIONS

To facilitate this study of children's verbalizations relative to the solving of mathematical problems, the following research questions were posited:

1. Do children, individually and in small groups, verbalize when problem solving?

2. If children do use language, how and for what purpose is the language used, relative to solving the mathematical problem, individually and in small group situations?

3. Is there any difference between the language and strategies used by an individual and a small group of mathematical problem solvers at each grade level as well as across the grades, grades two, three and four?

4. Is there a developmental pattern of growth and change in the mathematical problem solving procedures of children in grades two through four?

5. What is the role of teacher relative to the problem solving process?

PLAN OF THE STUDY

A brief overview of the plan of the study follows. A detailed explanation is presented in Chapter III.

The discovery-orientation of the study necessitated use of a research technique, exploratory in nature, for the purpose of gaining insights rather than verifying hypotheses. Intense observation by the researcher of individual and small groups was necessary in order to study if and how children use language while solving mathematical problems. A determined effort was made to intervene as infrequently as possible while the children were involved in the actual problem solving processes.

Twenty-six children in grades two through four were selected from the St. Albert Separate School System as subjects for this study.

The problem solving sessions were recorded by means of audiotape and videotape. The audiotapes were later transcribed and detailed examination of the data, using the protocols in conjunction with the audiotapes, was made.

In addition, a journal was used to record insights gained and inferences made relative to the use of language by children and the partitioning behaviors manifested.

REPORT OF THE FINDINGS

No predetermined classification scheme, based on expert opinion or upon existing plans, was used for analyzing subjects' responses. Instead, an attempt was made to be as true as possible to the experimental data in generating a classification scheme of the responses. Subsequently, the scheme which evolved for classifying subjects' responses was unique and relatively comprehensive.

LIMITATIONS OF THE STUDY

The limitations of the study are acknowledged as follows:

1. The limited sample size affects generalization of the findings.

Samples may or may not be representative of the manner in which all children use language when actively involved in mathematical problem solving or of the problem solving behaviors of all children at comparative grade levels.

2. Individual responses will vary in accordance with the experiential background of each child.

3. The exploratory nature of the study may delimit the findings.

SIGNIFICANCE OF THE STUDY

It is intended that this study will provide insight into children's language use and thought while solving mathematical problems. It is also hoped this study will serve as a basis for further examination of the role of oral language in elementary school mathematics.

OVERVIEW OF PLAN FOR REPORTING THE INVESTIGATION

The report of the investigation is presented as follows:

1. Chapter II contains a review of relevant literature and related studies.

2. Chapter III describes the design of the study and the derived theoretical framework used for analysis of the data.

3. Chapter IV presents a description of the major findings of the study.

4. The final chapter, Chapter V, summarizes and draws conclusions relative to the major findings of the investigation, and states implications for education. The chapter includes the reflections of the researcher upon personal insights gained from this study of young

children solving mathematical problems, individually and in small group situations, and concludes with recommendations for further research.

Chapter II

THE BACKGROUND OF THE STUDY

INTRODUCTION

The purpose of this chapter is to provide a report of relevant research from which evolved a theoretical framework for an exploratory study of how language is used by young children involved in solving mathematical problems. First, the relationships between thought and language, learning and language, and experience and language were examined for the purpose of determining possible agreement as to the nature of the role of language in children's learning. Next, a review of relevant theory and research reports regarding the nature of problem solving, problem solving in mathematics and, in particular, the solving of partitioning problems was undertaken to provide guidance in the development of the mathematical problem solving tasks and for interpreting the results of the experimental study.

THOUGHT AND LANGUAGE

The apparent relationship between thought and language has prompted much speculation among investigators in child psychology. One of the first theorists to draw attention to the importance of language as a system of controls in the problem solving activities of children was the Russian psychologist, Vygotsky (1962, translation of 1932). Vygotsky views language as the prime organizer of cognitive complexity wherein the development and improvement of language is functionally related to and operant upon the development of cognition. According to Vygotsky, the child's intellectual growth is dependent upon his mastery of the social means of thought, that is, language. Vygotsky contends that the word functions first as an invitation to a concept but eventually the word comes to symbolize the concept itself as a synthesis of the child's accumulating experience. It is the "word" which masters and directs problem solving. Whenever abstraction occurs, potential concepts play a role in complex thinking. Analysis and synthesis are the main instruments of abstraction but, again, the facilitator is the word. The relationship from thought to word and from word to thought is a process involving continual movement both ways. Indeed, thought is not merely expressed in words but it comes into existence through them (Vygotsky, 1962, p. 7).

Vygotsky (1962) believes the process of thinking becomes inextricably interwoven with silent formations of language. Once language becomes a part of thinking, thinking becomes flexible, differentiated, objectified, and experience becomes communicable. His contention that thought appears to be needed for language and vice versa would imply that an examination of the language used by young children when solving mathematical problems would be of prime importance.

Cognitive Growth and Language

Piaget and his Geneva associates (Piaget and Inhelder, 1969) posit the primacy of cognitive growth as regards language development. Piaget's colleague, Hermina Sinclair-De-Zwart (1973) found knowledge of linguistic structures and terms, or the teaching of relevant verbal terminology, little influenced the performance of children on conservation tasks. It was her conclusion that language development can not hasten cognitive growth. For Piaget and associates, cognition at all developmental levels consists of actions performed by the learner. Actions are overt and physical at lower levels of development but with maturity actions become increasingly internalized until covert actions, i.e., verbal, symbolic, and formal operations, are the dominant processes of cognition. Verbal enrichment or sophistications do not accompany changes in cognitive structures. "Language is not enough to explain thought because the structures which characterize thought have their roots in action and in sensori-motor mechanisms that are deeper than linguistics" (Piaget, 1969, p. 98). Piaget suggests that language may be structured by logic but is not the source of logic. Despite this position, Piagetians do not completely dismiss the importance of language for they concede once the structures of thought are refined, language becomes necessary for their elaboration and mobility (Piaget, 1970).

Although the value of language for the development of mental operations and cognitive structures is seriously questioned by Piagetians, not everyone agrees. For example, Donaldson (1978) contends that there is much convincing argument that a child's language learning skills are not isolated from the rest of his mental growth. From the evidence she has studied, Donaldson affirms the following major positions:

1. Children are not at any stage as egocentric as Piaget claims;

2. Children are not as limited in their ability to reason deductively as claimed by Piaget and others—spontaneous behavior evidences this to be so; and

3. A child's ability to learn language is not separate from but a part of mental growth.

The contentions of Donaldson seem to suggest that an examination of the language used by young children could possibly provide some insight into the thinking strategies employed in problem solving situations.

Donaldson's position was an outgrowth of Bruner's conceptualization of cognitive development as the ever increasing internalizations of technologies from the culture.

Language and cognitive development were viewed by Bruner as "technologies" to deal with the world (Bruner, 1964). Bruner and colleagues emphasize language as the most effective technology available. According to Bruner, the world can be represented in three modes: enactive (through actions), ikonic (through images), and symbolic. All three modes remain in the system throughout life, are sequential in development, and are interacting. In Bruner's terms, when higher level learning takes place, a transition from ikonic to symbolic representation occurs and it is brought about by language, a means to organize and integrate experience.

> Once the child has succeeded in internalizing language as a cognitive instrument, it becomes possible for him to represent and systematically transform the regularities of experience with far greater flexibility and power than before. (Bruner, 1964, p. 4)

Bruner's belief that language is a powerful cognitive instrument strongly suggests that to study the use of language by children as they solve problems would provide insight into how children think when engaged in a problem solving process.

Other Relevant Views

Other writers have developed theories which seem to support those of Bruner and Donaldson. It is Moffett's (1968) observation that the function of "abstraction" is a process underlying all stages of information processing from sensori-motor and perceptual to affective and intellectual (p. 19). Abstraction by selecting and ranking the elements of experience reduces reality to manageable summaries. To abstract is to trade a loss of reality for a gain in control (p. 23). This gain in control is evidenced in the child's language. Therefore, the child's thinking may be revealed through his language.

Schmidt (1973) states that the language used in communication is the same language used in thinking; therefore it is neither a means of communication nor an instrument of thought; it is both. Even so, this tempered position implies that the role of language in problem solving is both communicative and cognitive; hence language can be studied to determine possible developmental levels not only regarding language use but also to examine the development of thought or thinking strategies.

Further evidence ligating language and thinking can be found in the intercorrelation tables of the WISC-R (Wechsler, 1974). Vocabulary, a highly verbal subtest almost consistently correlates higher with total ability scores than any other subtest (pp. 36-47).

In summation, the theorists view language as an essential part of thinking and of problem solving. The specific role of language, however, has not yet been determined and further study would appear to be warranted.

LEARNING AND LANGUAGE

Language is also seen as a tool by which children make sense of their learning, re-interpret knowledge and examine existing assumptions while developing a representation of the world (Britton, 1970). A child's language power reflects the way he is experiencing the world and determines how he operates in it (Britton, 1970, p. 2). The child's cognitive functions are his means of making sense of the world, of current and past experience and his language is the "key system" whereby "he represents the world to himself and organizes all other ways of representing" (p. 21).

Barnes (1972) contends that the very act of verbalizing requires a re-organization of the old and the new together. Barnes further states that part of good teaching is in perceiving how language used by children contributes to or inhibits learning. In his writings, Barnes purports that not enough is known about: (1) how various uses of language function as a means of learning and (2) the extent to which these functions are determined by the pupil's perceptions of the content for that language.

Piaget, from his examination of children's talk, proposed a broad classification of language functions which has provided a model for later work that takes a functional approach (Piaget, op. cit.). In major ways, he sees young children's language as 'egocentric,' that is, fulfilling self needs and failing to accommodate the views of others. The remainder of children's speech, Piaget classifies as 'socialized' and as realized in requests, demands, answers and adapted information. The use of such categories establishes linguistic structures as criteria for classifying but fails to recognize that speech in the same form can be directed towards achieving different goals; for example, a demand may serve to retrieve property or as an imaginative pursuit of a play sequence to project imagination into experience.

Halliday (1969) poses a sociolinguistic view of the functions of children's language to language itself. He states, "Language is, for the child, a rich and adaptable instrument for the realization of his intentions, and there is hardly any limit to what he can do with it" (p. 27). Halliday proposes seven functions or ways the child uses language to make meaning (1977): (1) instrumental or "I want" through which the child's material needs are met; (2) regulatory or "Do as I tell you" through which the child gets others to do as he wants; (3) interactional through which the child interacts with someone else; (4) personal through which the child expresses his uniqueness and self-awareness; (5) heuristic or "tell me why" through which the child explores the environment; (6) imaginative or "let's pretend" through which the child creates his own environment; and (7) informative or representational, e.g., "I've got something to tell you" through which the child conveys information to someone. Halliday identifies an orderly emergence of the functions of language used by children to establish meaning during the lifetime of a child, progressing from approximately age nine months to three and a half years. This seems to indicate that children of school age should be given many opportunities to use language to learn. Halliday, however, does not identify specific strategies embedded within the language functions. The language uses classified by Piaget as 'adapted information' and by Halliday as 'informative or representational' remain undifferentiated although recognized to serve different uses.

Tough (1969) sought some means of classification within the broad category of informative language with which to differentiate the purposes for which children can and do use language. Examination of children's language led Tough to regard functions "as being the characteristic modes in which language is used to organize or order experiences and intentions" (Tough, in preparation, p. 9). The modes are inferred from the evidence of particular strategies selected by the child for conveying meaning. Tough contends each utterance serves at least two different kinds of functions; a relational function and a range of content or ideational functions. The relational function seldom operates independent of the ideational functions, the exception perhaps being greetings. From the language of three year olds which had been collected in play situations, and their language at five and seven years old which was collected in task oriented contexts, Tough identified the following ideational functions: (1) the self-maintaining function, (2) the directive function, (3) the interpretive function, (4) the projective function, (5) the predictive function, (6) the imaginative function and (7) the empathetic function. Embedded within each function, Tough perceives thinking strategies relative to but with considerable variance from context to context. As do Britton, Barnes and Halliday, Tough (1978) views language as a means to represent the world and as a means to co-ordinate or regulate actions.

> The language a child has developed serves as a means of communicating about his thinking, and of developing his own and other people's actions; at the same time the language used by others with him helps him to find order, significance, and meaning in the world around him and to establish values for different activities and experiences. (p. 28)

Growth in language power is essentially related to the individual's functioning along the concrete-abstract continuum in thought and the implicit-explicit continuum in language production.

Tough's broader view of the ideational functions of language and the thinking strategies offers a possible approach to or basis for examination of children's language used in mathematical problem solving.

LANGUAGE AND EXPERIENCE

Language helps children make sense of their experience (Lindfors, 1980; Smith, 1975; Britton, 1973). Such a statement leads to the question "Can language contribute to cognitive growth?" which is raised by Smith (1975).

Basic to Smith's approach is the assumption that every human builds out of personal experience, a cognitive structure or "a theory of the world in his head" (1975, p. 11). This personal theory of what the world is like shapes both the way past experience is recalled, summarized, interpreted, and the way new experience is comprehended. Smith also contends that every human endeavors (1) to comprehend, that is, make sense of the world by relating new experience to the already known and (2) to learn when experience does not fit our 'theory' by altering existing cognitive structures. He perceives that as children endeavor to comprehend and learn, the language of the children serves to question, focus attention, precise understandings, make understandings retrievable, reinterpret experience and to go beyond present personal experience. Based upon his research findings, Smith claims language, though not the only tool, is a powerful tool to comprehend and learn in order to make sense of one's world.

Like Smith, Britton (1973) views language as a tool to make sense of experience. He writes that a child begins "with a drive to explore the world he is born into" and that speech early becomes the "principle instrument" of exploration (p. 93).

Lindfors (1980) supports the view that language helps children make sense of their experience. She contends that their theory of the world grows and changes as children encounter other's experiences, ideas, and interpretations and "this encounter happens most of the time through language interaction."

As a means of reinterpreting past experience language also serves as an aid. Britton (1973) writes:

> Language is one way of representing experience . . . we habitually use talk to go back over events and interpret them, making sense of them in a way that we are unable to when they were taking place. (p. 19)

He views language as a means by which we re-present past experience, present, consider, and re-interpret it again in terms of the ever-changing and growing "theory of the world" which shapes the interpretation, i.e., comprehension, but which is also shaped and modified by experience, i.e., learning. Britton speaks of "symbolizing reality by means of language in order that we may handle it [reality]" (Britton, 1973). Thus, language provides one important means of comprehending and learning.

Tough (1977) writes of non-interactional oral language serving as an aid to focus attention. To dismiss the simple monitoring language use would be, in Tough's opinion, to ignore how important activity is to the building of a theory of the world and, at the same time, to dismiss action related use of language as an important aid to comprehending and learning. In other words, children's own talking, as well as the talk they interact with, can help focus attention upon the comprehension and learning of an immediate task.

With respect to mathematics, learning of labels is also related to experience. Early studies indicate language labels aid recall, making understandings more retrievable (Carmichael, Hogan and Walter, 1932; Brown and Lenneberg, 1954; Frijda and Van de Geer, 1961). To 're-collect' experience, the encoding of the experience in language, the talking about it, makes an experience more readily retrievable and the way the experience is labelled will influence the recall, that is, the labels themselves become part of the remembered experience.

Piaget contends a major aspect of cognitive growth, egocentrism, relates to the child's perception of experience being limited to a single perception, his own. It may be that in moving the child to a more variable point of view, that interaction makes its greatest contribution to cognitive growth.

Donaldson (1978) writes:

> Children are able to learn language precisely because they possess certain other skills—and specifically because they have a relatively well-developed capacity for making sense of certain types of situations involving direct and immediate human interaction. (pp. 36-37)

Thus it would seem that children use the understood human situation to predict language and social behavior and to eventually learn to solve abstract problems. Donaldson believes that it is vital to be able to eventually operate with abstractions in order to manipulate variables, see new possibilities, solve problems, in brief to control one's thinking. She quotes Vygotsky: "All the higher functions have in common awareness, abstraction and control" (1978, p. 99). For young children, language does not exist in an abstract sense independent of the objects and ideas for which it stands. Therefore Donaldson perceives the role of the teacher as necessarily organizing well-understood personal experience of children in a way that makes its structure clear, thereby enabling the child to focus on meaning and to gradually become free of context. Thus interaction with understandable tasks enables children to comprehend and learn beyond the task itself, that is to reshape, rethink and expand experience.

The notion of language aiding in going beyond the present situation is a major tenant of Bruner (1964) who views language as a cognitive instrument whereby children represent, manipulate and transform "the regularities of experience" apart from the direct experience itself. Perhaps Bruner means that by language becoming a "cognitive instrument" for children, the children move from language that relies heavily on personal, present, direct experience toward a greater independence of language from immediate situation. Bruner writes further:

> As for how language becomes internalized as a program for ordering experience my speculation is the process depends upon the process of interaction with others. (Bruner, 1964, p. 166)

He stresses the reciprocity in learning, i.e., the way we learn in dialogue and perceives dialogue as the ideal circumstance for internalizing because of the process of interaction with others. Bruner argues for greater emphasis upon spoken language in learning and in doing so aligns himself with Vygotsky.

Moffett (1968) also acknowledged the dependency of language upon mental growth as did Vygotsky. According to Moffett, the major dimension of growth is that the self enlarges, assimilating the world to itself and accommodating itself to the world. In order for the growth to occur, however, interaction is essential. Thought and speech must be matched and the closer the match the better the communication. Oral language matches thought with speech, and one's own mind with the minds of others. Moffett contends that the truly basic skills of thought, making messages and making sense of messages from others are developed at the levels of experience and oral speech. Moffett and Wagner (1976) write that "without the level before, the next level is impossible." In the process of talking about experiences and perceptions, and in listening to others talk about theirs, children learn to modify or expand their own internalized structures as well as to communicate them without ambiguity. Verbal interaction in an environment of common experience is for Moffett a time for honing thought and language. From the work of Moffett and Wagner, two major implications for the teaching of mathematics can be deduced. First that

> Teachers of mathematics like teachers of language, have mistakingly assumed because performance measures are based upon skill in reading and writing numerally coded information, that these are the basic skills. They are not. They are derived skills and impossible of meaningful acquisition unless grounded in the basic skills of thought and oral verbalization about mathematical ideas. The teaching of mathematics should focus on thinking, to match thought with speech, and verbalizing, to match mind with mind.

Secondly:

> Mathematic teaching and learning experiences should be symbolized in the verbal codes of ordinary language at both the oral and written levels before superimposing the coding scheme of mathematical notations. (Skypek, 1981)

PROBLEM SOLVING

It is possible that 'problem' used in the general sense of achieving a desired state, in which case any thinking with a desired result, can be considered to be 'problem solving.' In this narrow sense problem solving does not include the concept of 'understanding' or 'clarifying a situation' (de Bono, 1980). Moreover, de Bono contends that too often understanding and clarifying are considered to be processes of perception and thinking is then regarded as the process of working upon perceptions to solve a problem. de Bono defines thinking as the "deliberate exploration of experience for a purpose" and that purpose may be "understanding, decision making, planning or problem solving." Thinking is the operational skill through which intelligence acts upon experience. Thinking, according to de Bono arranges and rearranges perception in order that children may have a clearer view. "The job of thinking is to clarify perception."

Problem Solving and Mathematics

Problem solving in mathematics is viewed as the "process of applying previously acquired knowledge to new and unfamiliar situations" (National Council of Supervisors and Mathematics, 1977). It may also be an individually acquired set of processes brought to bear on a situation that confronts the individual (Le Blanc, 1977). It would seem that the latter definitions imply the previously stated ideas of de Bono regarding experience, intelligence and thinking as a process of clarification.

The terms problem and problem solving are further defined by Lester (1980) as:

> A problem is a situation in which an individual or group is called upon to perform a task for which there is no readily accessible algorithm which determines completely the method of solution" (Lester, 1978, p. 54). It follows that problem solving is the set of actions taken to perform the task (i.e., solve the problem). This definition is consistent with definitions presented by several others (e.g., Bourne, Ekstrand & Dominowski, 1971; Brownell, 1942; Duncker, 1945; Henderson & Pingry, 1953; Kinsella, 1970; Newell & Simon, 1972; and Resnick & Glaser, 1976).

Seemingly, the above mentioned "set of actions to perform the task" would depend upon thinking, that is, exploration of experience for a purpose.

Polya and Problem Solving

Polya (1965), the mathematician, identifies four procedures inherent in problem solving: (1) understanding the problem, (2) planning to solve the problem, (3) solving the problem, and (4) reviewing the problem and the solution. Though having identified the necessary steps of the problem solving process, Polya's heuristic approach does not identify within each step those "processes" or "set of actions brought to bear upon the task," that is, Polya fails to examine "the exploration of the experience."

Developing Problem Solving Ability in Mathematics

Children and Problem Solving

In developing children's problem solving ability in mathematics, it is vitally important to identify problems appropriate for children of differing ability and in different grade levels. The research of Kieren (1980, 1979) and Pothier (1981) examines the problem solving achievements of young children seeking solutions to problems which embed the rational number construct.

After delineating the cognitive aspects of rational number, Kieren (1976, 1980a, 1980b) concludes that the system of rational numbers is a complex construct arising out of real-life situations. Basic to the rational number construct are partitioning and equivalence

(Kieren, 1980c). Pothier (1981) examines the partitioning mechanism as it relates to rational number construct. She sought not only understanding of rational number development but also the implications for the scope and sequence of a primary mathematics program.

Pothier examined the solutions achieved by young problem solvers to determine what children know, that is, the focus appears to be upon the achievements rather than upon how children come to know using the processes apparent in the problem solving steps identified by Polya.

For the present study, the focus is on how children come to know as they solve partitioning problems. The role of language in this important facet of mathematical learning needs to be identified and clarified.

LANGUAGE AND MATHEMATICS

Among the first to be concerned with the relationship between arithmetic and language was Thorndike (1912). In speaking of the measurement of academic achievement, he stated, "As you will know our measurement of ability in arithmetic is actually a measure of two things: sheer mathematical insight and knowledge on the one hand; and acquaintance with language on the other." It is highly probable that Thorndike's concern for children getting meaning focused on the word rather than on the broader goal of language use for learning which is necessarily more than acquaintance with language. However, almost seventy years later, there is little reported research to support the latter portion of Thorndike's statement, that is, acquaintance with language. It appears the role of language within the context of mathematics has rarely been examined.

According to several reported findings, a number of language factors correlate with success in mathematics. Linville (1969) concluded that both syntactic structure and vocabulary level, with vocabulary level perhaps more crucial, are important factors in the ability to solve verbal arithmetic problems. The research reviewed by Kirkpatrick (1970) in the area of vocabulary indicates a consistently positive and strong connection between mathematics, especially problem solving, and mathematical vocabulary (Buckingham, 1937; Eagle, 1948; Foran, 1933; Johnson, 1944, 1949; Lyda and Duncan, 1967; Vanderlinde, 1964). This is not surprising since vocabulary knowledge is basic to comprehension and students must comprehend the problems before they can apply the mathematical concepts necessary for the solutions (Kirkpatrick, 1970).

Brindley (1980), in a study of ratios and fractions, reinforces the concept that talk supports learning. The results of the study substantiate what has not yet been supported by the evidence—the belief that language and talk are an inextricable part of learning. Knight and Hargis (1977) contend that children's language development is likely to affect their mathematical learning. For example, mastery of the grammar of one-to-one correspondence leads to the concept of "manyness." A second set of grammatical patterns occurs in noun phrases that "contain the fundamental language vehicle for presenting arithmetic concepts." Finally, Knight and Hargis state an understanding of the syntax of comparative construction is essential to coping with arithmetic reasoning problems. In a previous study, Hargis (1976) found that a significant number of "normal" children do not have adequate language mastery for success in mathematical settings. If many "normal" children do not have proper language structures for mathematics, then it seems highly probable that the inference can be drawn that an even greater number of children learning English as a second language and linguistically handicapped children may have similar deficiencies.

According to Corle (1972), research has shown that knowledge of mathematical concepts alone will not guarantee success in mathematics, particularly in the area of written or verbal mathematical problems. The evidence from the cited studies strongly suggests that students' success in mathematics seems inextricably interwoven with their level of language power. The latter can be considered the integration of two major areas of the child's development, his thinking power and his language competence (Siemens, 1973, p. 9).

Reading and Mathematics

Studies into the relationship of reading to mathematics date back to the 1920's and 1930's. Kirkpatrick (1970) cites Stevens' (1932) conclusion that ability in fundamental operations of mathematics was more closely correlated with ability in problem solving than with general reading ability. This finding was confirmed by Morton (1953) who reported that skill in problem solving correlated highly with skill in fundamental operations and intelligence, but showed a low, though positive, correlation with reading speed. Other studies in this area do not necessarily agree with this conclusion (Cummins, 1982; Martin, 1964; Pitts, 1952; Skillman, 1972).

Although research is not conclusive, there appears to be general agreement between mathematics educators and reading educators on skills of reading mathematics that are common to other content fields as well (Morgenstern, 1969; Muelder, 1969; Niles, 1969; Weber, 1962). In addition to the reading skills which are operative in the discipline of mathematics and common to other content areas, research findings support the need for specific reading skills, for example, adjustment to vocabulary, adjusted rate of reading, reading charts and graphs, and reading in verbal problem solving. Not only are specific reading skills required by the reader of mathematics, the unique vocabulary, words, and symbols as well as the terse, concise nature of the material must be considered (Aiken, 1971; Clark, 1969; Cuevas, 1984; Heddens and Smith, 1964; Kerfoot, 1961; Muelder, 1969; Reed, 1968; Stauffer, 1966).

However, there seem to be matters even more critical than the sheer ability to decode and comprehend printed material that relate to the use of language in mathematics. Referring to young children solving problems in mathematics, Nelson (1980) contends that the spontaneous behavior of children solving problems is varied and seems to serve different functions at different age levels. Based on his research findings, he concludes that the whole area of verbalizations and their role in problem solving needs further study (p. 19).

SUMMARY

Readings related to language and thinking show that the two processes are so interwoven that separation would be very difficult. As a consequence, mathematical thinking would likely contain a large language component. This would not seem to be supported by research and theory. Therefore, there is a need to examine the role of language when children think through to solution given mathematical problems. The conclusion reached at the Commonwealth Association for Science and Mathematics, Accra, 1975, seems pertinent almost ten years later in that, "Any attempt to compile writings regarding the role of language and mathematical education can only indicate the state of our ignorance." The means to help alleviate such ignorance will be sought by way of studying children while they strive to solve partitioning problems in mathematics.

Chapter III

THE DESIGN OF THE EXPERIMENTAL STUDY

The purpose of this chapter is to present the methodology of the study. The chapter includes information concerning selection of the sample, the research instrument employed, methods used for collection of the data, and the treatment of the collected data.

SELECTION OF THE SAMPLE

The subjects of the study were selected during the fall of 1982 from regular classrooms, grades two through four, in a school in the St. Albert Separate School System. Access to the school and to the students was made available because of prior teaching experience within the school by the researcher. Children assessed to be of at least average intelligence, average or high language users, and capable of interacting in the problem solving sessions were chosen. The selection was based upon the observations and experiences of the regular Language Arts teachers who worked with the children on a daily basis. Initially, two individuals and two small groups of three were chosen to represent each grade level. However, due to the mass of data derived for analysis, only one individual and one small group of three were selected from each grade level for analysis though protocols for all are available.

Letters were sent by the researcher to the parents of the subjects (see Appendix A for a copy of the letter) which outlines the nature of the study and sought parental permission to include the data derived from the children who partook in the study.

RESEARCH INSTRUMENT

The research instrument used in this study was adapted by the investigator from the original tasks designed by Kieren. The discussion of the research instrument provides further description of the tasks and the necessary adaptations of the tasks for this study.

The original partitioning tasks, prepared by Kieren (1979) for students of grade four, as part of a written test in mathematics, were based upon real life experience, the sharing of pizza. In the spring of 1982, the tasks were piloted in oral problem solving situations, first with children in grade four, then with children in grade three and lastly, with grade two children. At each grade level necessary alteration of the tasks to make "a match" with the children became apparent. The alterations made were:

1. At the grade four level, the space between the picture referring to the girls and the picture referring to the boys was widened. This seemed to separate the two steps of the problem.

2. At the grade three level, the original question, "If each child gets a fair share of the pizza, who would get more pizza, each boy or each girl?", was perceived as too encompassing and was preceded by two questions: (a) "How much pizza does each boy get?" and (b) "How much pizza does each girl get?" The difficulty experienced answering the original question was eased by the revision.

3. At the grade two level, it was recognized that the question, "How did you figure this out?" would require the children to talk the answer through as their writing skills were limited.

Although the tasks were not further revised, during the actual investigation, it became apparent that to separate the drawings of the pizza from the caricatures would have possibly prevented the children perceiving the pizza as belonging to the "child because it's on his head."

These minor revisions appeared to reduce the task to a more understandable, hence more manageable, assignment even though no manipulatives were provided. Moreover, the task itself tended to be abstract, especially for grade two children.

Each of the following five questions was posited below each of the eight possible tasks:

1. How much pizza does each boy get?
2. How much pizza does each girl get?
3. If each child gets a fair share of the pizza, who would get more pizza—each boy or each girl?
4. Does each boy get as much pizza as each girl?
5. How did you figure this out?

A copy of the eight tasks presented in sequential order is provided in Appendix B.

COLLECTION OF THE DATA

The data for this study were collected between October 15 and November 30, 1982.

In order that the children remain in their classrooms during 'prime teaching time,' the data were collected in the afternoons. The tasks were presented in sequential order to the individual or small group. No time limits to complete the tasks were specified. Instead, the students determined when they wished to terminate the activity. Sequential interviews were arranged. Teachers were most co-operative in releasing subjects from their classes as requested. No subject was required to forego his/her favorite subject or activity for an interview and an effort was made to create minimal disruption of regular classroom routines.

The data collected were video and audio tape recorded and later transcriptions of audiotapes were transferred to typewritten protocols. Samples are provided in the Appendix.

The Data Situation

Throughout the time of data collection the researcher endeavored to not intervene in the mathematical problem solving processes. To be consistent, the first of the eight problems was introduced as follows: "I have eight problems about sharing pizza. I would like to give you the first problem and have you decide how you would share the pizza fairly among the boys and among the girls. Please share the pizza fairly. You are the boss, the pizza cutter and you know everyone likes fair shares. Answer the questions and when you are done I will give you problem two."

As the children worked to solve the problems, the researcher made the pretence of "sorting junk" or marking papers. In fact, an effort was being made to observe and record in a log book the language and mathematical behaviors of the children.

When it appeared the children might seek more than minimal direction, the researcher left the room but endeavored to attend to the happenings by listening through use of the intercom.

Questions and comments directed to the researcher by the children were typically responded to by further questions or probes, for example, "What do _you_ think?", "Can you tell me more?", "What makes you think so?" or "Could you try and think of a different way?" Though not always possible, the aim was to interact with the subjects when necessary but not to teach or to direct the achievements.

Method of Data Collection

The problem solving sessions were recorded on VHS tapes, of two hour length, using a Panasonic Video Cassette Recorder (NV-8200) and a Panasonic Video Camera (WV-361P). As well, the sessions were recorded on a Panasonic Audio Recorder (RQ 2133) which was situated on the table to the left of the children. The majority of the children seemed to forget that they were being videotaped as they became involved in the partitioning tasks.

To supplement the tapes, intense observation of the children was made during the problem solving sessions and the observations were recorded in a journal, noting the insights gained and inferences made pertaining to the children's language use and the partitioning behaviors manifested.

TREATMENT OF THE DATA

No predetermined plans for analysis and treatment of the data were made. Instead subjects' responses provided direction for devising procedures and techniques most appropriate for analyzing the protocols and reporting the findings. Information gathered from the literature, particularly the writings of Tough and Polya, provided valuable insight that gave rise to the criteria established for analysis of the data. Derivation of a system appropriate for categorization of the children's language was accomplished through use of the following procedures:

1. Intense examination and observation of the videotapes to determine patterns of behavior, verbal or non-verbal.

2. Comparison of the protocols of the transcribed audiotape with the videotape, again searching for patterns of behavior, verbal or non-verbal.

3. Examination of the protocols and videotapes using Polya's problem solving framework.

4. Examination of each of Polya's problem solving steps determined in the protocols using the oral language functions and strategies proposed by Tough.

5. Elimination of those oral language functions and strategies determined by Tough which did not appear in the data.

6. Addition of recurring oral language functions and strategies appearing in the data, beyond those proposed by Tough.

Reliability of Classification of Students' Oral Language Responses

The reliability of the classification of the types of student responses was established by inter-scorer agreement.

The percentage of agreement between investigator and an independent judge was calculated for each protocol. The person involved in establishing the reliability of the classification of student responses was an experienced classroom teacher and as well an experienced reading clinic diagnostician formerly with the Ottawa Public School System.

The percentages of agreement between investigator and judge in relation to the protocols of the independent subjects were: grade two—93.3%; grade three—95.7%; grade four—94.8%.

Similarly the percentages of agreement between investigator and judge in relation to the protocols of the small group were: grade two—90.5%; grade three—94.2%; grade four—96.5%.

SUMMARY

In this study, designed to explore the function of children's oral language use in solving mathematical problems, specifically partitioning problems, in grades two, three and four, introspective techniques were used in order to examine the language and thinking processes. During individual and small group problem solving sessions, a maximum of eight partitioning problems were presented in sequence. As much as possible, the researcher endeavored not to intervene in the problem solving activities.

All responses were video and audiotape recorded, transcribed as written protocols and analyzed by the investigator. Criteria for analysis of the data were determined by the nature of the subjects' responses and based upon reported research and theory related to children's use of language and the problem solving process. Treatment of the data was primarily descriptive.

In the next chapter, Chapter IV, criteria established to make the analysis possible will be presented.

Chapter IV

ANALYSIS OF DATA AND REPORT OF FINDINGS

The purpose of this chapter is to present the major findings of the study which sought to examine how children use their language to facilitate problem solving in given mathematical tasks.

To attain this goal it was desirable to select children recognized to be highly verbal and of at least average intelligence. Observation made by classroom teachers of children in grades two through four determined the children who met the two requisites; verbosity and at least average intelligence.

The pizza problems proved to be highly effective in enabling small groups of children to demonstrate partitioning abilities while engaged in dialogue. Also, individual children chosen to perform this task were highly verbal in the process, thereby making it possible to pursue and study the language used by individuals to facilitate their thinking and mathematical learning as well as to examine the language used by children working within small groups.

The descriptive data and findings relate to the use of language by young children while engaged in mathematical problem solving, specifically, partitioning on circular areas.

The problem solving processes of one individual and one small group from each grade, two through four, will be described. At the conclusion of each problem solved, a summary of the language-thinking and mathematical-thinking is provided to indicate the pattern of development expressed by the individual and the small group from each grade level.

As previously stated in Chapter III, no predetermined criteria were available to allow analysis of the experimental data. Derivation of criteria arose through use of Polya's framework for mathematical problem solving and examination of the language used by the children manifested within the framework. This procedure enabled the researcher to identify specific language functions and strategies used to facilitate the mathematical learning. The plan to examine the language functions and strategies arose from study of the work of Tough who initially examined language arising in children's play. The methodology of Tough in conjunction with the framework of Polya enabled the researcher to derive criteria which made possible the perceptions which have been drawn from selected excerpts found in transcriptions considered most pertinent to children solving given mathematical problems.

Included in this chapter, in sequential order, is the following information:

1. An overview and outline of criteria derived for analysis of the data.

2. Interpretation of data collected from (a) an individual grade two child and (b) a small group of grade two children.

3. Interpretation of language and learning strategies arising from data collected from individual children and small groups of children in grades three and four.

4. Knowledge of mathematical labels.

5. Non verbal behavior observed while children solved the partitioning problems.

OVERVIEW OF CRITERIA USED TO ANALYSE THE DATA

Since the given learning tasks were mathematical, Polya's framework for problem solving served as a basic framework for the analysis of the data. Elaboration of each step, made possible from careful study of children's verbal and non-verbal behaviors derived from selected audio-visual transcriptions, evolved. As patterns of children's language use emerged, it became increasingly evident that four major functions and strategies identified by Tough with the addition of one important overriding category, the clarifying function, would serve as appropriate categories for analyzing and describing the language and learning strategies applied while solving the mathematical problems. To facilitate understanding of the criteria applied in the analysis of the data, a summary of the five main language functions precedes the outline of the criteria used to analyze the data.

Summation of Language Functions

Clarifying Function

Clarification is concerned with successful interpretation and understanding. This function relates to concern with status of self in that the desire to succeed is integral and also to other recognition as approval, agreement and reassurance are sought.

Reporting Function

This function relates to immediate and recalled experiences. The purpose seems to be one of identification of the elements of the experience and is served by a number of strategies which move toward logical reasoning in which the experience is organized at different levels of meaning.

Directing Function

This function is concerned with directing the actions of self and/or others.

Self-Maintaining Function

This function is concerned with the comfort, feelings, success and status of the self and with preventing trespass on one's property, person and rights. Maintenance of the self necessarily involves trying to control others and to not be controlled by others and hence is realized in reaction to others. Self-maintaining and relational functions of language are not independent of one another.

Predicting Function

This function is concerned with projecting beyond the immediate experience, a seemingly cognitive ordering of events which have not yet happened.

OUTLINE OF LANGUAGE FUNCTIONS AND STRATEGIES WITHIN POLYA'S PROBLEM SOLVING FRAMEWORK

I. Understanding the Problem:

Language and strategies used by the learner were:

1. Clarifying the Meaning of the Stated Problem
    a. interpreting the problem by reading aloud to the self
    b. interpreting the problem by reading aloud to others
    c. assuring the interpretation

        - i. seeking agreement of understanding
        - ii. justifying the interpretation
        - iii. surveying possible alternatives
- 2. Clarifying the Meaning of the Illustration
    - a. interpreting the illustration
        - i. directing the self
        - ii. monitoring detail
        - iii. comparing details
    - b. reporting
        - i. observations
        - ii. analysis
        - iii. procedures to interpret
        - iv. conclusions
    - c. imagining by developing an imaginary situation based upon real life

II. Planning the Solution

Language was used for:

- 1. Reporting
    - a. toward logical reasoning
        - i. referring to detail
        - ii. recognizing related aspects
        - iii. making an analysis using the above
        - iv. recognizing and extracting central meaning
        - v. reflecting on the meaning of the experience including own feelings
        - vi. justifying judgments and action

vii. reflecting on events and drawing conclusions

viii. recognizing solutions

ix. sequencing activity

2. Predicting
    a. anticipating possible solutions
    b. anticipating and recognizing alternatives
    c. predicting consequences of actions
    d. predicting alternate solutions based upon imagined but real life situations
    e. anticipating a sequence of events
3. Directing
    a. monitoring one's own action
    b. directing the actions of self
    c. directing the actions of others
    d. questioning to seek collaboration
    e. collaborating with others
4. Self Maintaining
    a. identification of self interest
    b. justifying and recognizing ownership
    c. establishing conditions
    d. surveying possible alternatives

III. Solving the Problem

Language was used for

1. Directing by
    a. monitoring self or others
    b. guiding actions of self or others
    c. collaborating with others

2. Reporting—moving toward logical reasoning by
    a. explaining a process
    b. explaining a process accompanied with illustration
    c. justifying judgement
    d. reflecting upon action and drawing conclusions
    e. recognition of principles and concepts
    f. analyzing problems and solutions
3. Clarifying by questioning in order to
    a. seek assistance
    b. seek approval
    c. extract meaning

IV. Reviewing the Solution:

Language was used for

1. Clarifying ideas by
    a. questioning
    b. reporting—moving towards logical reasoning
        i. explaining
        ii. justifying
        iii. comparing
        iv. reflecting
        v. concluding
        vi. recognizing principles
    c. evaluating
    d. reading aloud
    e. directing
        i. questioning to seek collaboration
        ii. directing actions of self or other

SOLVING THE PIZZA PROBLEMS: GRADE TWO

Individuals and small groups were involved in solving the pizza problems. Therefore, discussion of the strategies employed by the individual child in attempting to solve the pizza problems is followed by a discussion of the small group strategies employed by the grade two children.

Problem Solving Strategies of the Individual Child

Since there were eight pizza problems to solve, the major strategies used by John in each situation, beginning with the simplest problem and progressing to the most difficult, will be described.

Pizza Problem One

Given the simple task of allocating one whole pizza to each boy and one whole pizza to each girl, John demonstrated that the following strategies were necessary.

I. Understanding the Problem. Two strategies were initiated by the problem solver when endeavoring to solve the problem. The first strategy was to read aloud to the self the five posed questions (stated in Chapter 3). Of interest while reading aloud to the self was the interruption by comments (e.g.,"Pizza -- right -- it's confusing. Well I think so -- if it's fair shares"). This seemed to be an acknowledgement that the problem had been read correctly, leading to the conclusion that the problem was confusing. When attempting to understand the problem, it seems possible that the child's concepts of 'fair' and 'share' evoke confusion. This notion is supported throughout the problem solving procedure and is demonstrated by the difficulty encountered by the problem solver in separating the two sub-problems, boys sharing pizza from girls sharing pizza. In order to achieve 'fair shares' there is the attempt throughout the problem solving process to solve one all inclusive problem, children sharing pizza, rather than to solve each of the two sub-problems. The confusion, giving rise to the difficulty described, is first identified in the commentary of the child while reading aloud to the self, the first strategy employed.

The second strategy was to read the same questions to another, the researcher, as though reassurance was being sought. No commentary accompanied the reading.

Strategies to clarify the understanding were reading aloud to the self and reading to another. Meaning was seemingly brought to the problem by use of these strategies as the child then was able to proceed to plan to solve the problem.

II. Planning to Solve the Problem. By means of a question, John sought further clarification regarding the precise activity needed to solve the problem (e.g., "Do I just draw in the pieces here -- to make fair shares"). The implied reference of the researcher to the original questions (i.e., "Maybe you had better do the boys first") was recognized and reported by John (e.g., "That's a question").

III. Solving the Problem. To solve the problem, the child used his language to direct the action (e.g., "He gets four pieces"), to monitor the action (e.g., "Okay. Just cut them like that. Okay") and to report the progress (e.g., "Three" and "That's her pizza"). In response to the researcher's probe, "Can you tell me why you're dividing them up?", the problem solver reported, "I'm dividing them up _so_ each boy and each girl gets -- like -- 4 pieces. _Cause if_ I was dividing them up into 4 over here and 3 over here it wouldn't be fair shares." This report implies more complex thinking than simple monitoring or directing of one's own actions. The reason stated implies a move toward logical thinking to explain the process and justify the behavior.

The functions of the language used to solve the problem were directing (the strategies being to direct the action of self and to monitor the action) and reporting. Embedded in the report is an analysis which refers to the components of the problem and a reflection upon the meaning of the solution. Evidenced by the solution, the problem solver realized that each boy and each girl would receive one whole pizza but 'fair shares' seems to imply equal sized pieces.

IV. _Reviewing the Problem and the Solution_. The solution reported was "four quarters." In response to the query, "What's another name for four quarters? How much has he got?", the problem solver stated, "four pieces -- that's one whole pizza." The child determined that "four quarters" is "four pieces"; that is, "one pizza." One pizza is recognized as "one whole piece" and it is restated that in this problem, one whole piece seems to be "actually one whole pizza." Language served to report and perhaps justify the findings. Furthermore the child's talk suggests that the problem solver had reflected upon the solution, made an analysis, drawn conclusions and evidenced a determined effort to synthesize the conclusions.

Summary: Language and Strategies Used in Solving Mathematical Problem One. Once the problem was clarified and the solution planned (i.e., "Do I just draw in the pieces here, to make fair shares?"), the problem solver had no difficulty and began partitioning in fourths using a horizontal line, halving, and its perpendicular bisector.

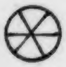
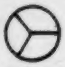
Pizza Problem Two

To solve the problem of sharing one pizza among three boys and three pizzas among nine girls, John demonstrated the following specific strategies were necessary.

I. Understanding the Problem. As the child used his language to clarify his ideas and seek understanding, three strategies were apparent: reading aloud to the self, reading to another, the researcher, and questioning. Having read, the problem solver concluded, "These are the same questions" and then queried "Fair shares?" as if to affirm the desired procedure.

II. Planning the Solution. The plan was evidenced in one utterance (i.e., "Make it equal to six pieces"). John used his language to direct the activity of self. This plan necessitated use of the halving mechanism to solve the problem and was initiated with a horizontal cut.

III. Solving the Problem. The solution of the problem was sought in two steps: (1) to share the pizza fairly among the boys and (2) to share the pizza fairly among the girls, using the same procedure as used in step one, partitioning to eventually achieve thirds.

During the first step, the language again seemed to provide self-direction and monitor the action (e.g., "-- across [] -- and three boys -- No -- eraser -- []"). In response to the probes, "What are you trying to make?", "Why?", the problem solver reported a perceived relationship to justify the activity (i.e., "Pieces -- so each boy gets one fair piece of pizza"). Talking to the self (whispering) accompanied the second step, partitioning in thirds (e.g., "-- like that --").

IV. *Reviewing the Problem and the Solution*. The problem solver used his language to clarify by questioning (e.g., "dividing sign?") and to try to logically reason the solution. Several attempts towards logical reasoning were apparent. To justify the judgment, "they get the same amount," the problem solver stated, "I needed three pieces *cause* there's three boys," "Cause each boy gets one piece of pizza out of three [pieces] *and* each girl gets one piece of pizza out of three [pieces]," and "I drew three lines *so* each girl got one piece out of three." To explain the process, John reasoned, "I figured it out by starting in the centre and drawing three lines out from it [the centre]" and "I divided the girls up into groups, three groups, this pizza, this pizza, this pizza." An effort had been made to compare the two solutions: "I took -- took them out of one pizza and put it with the other and the other." The problem solver had reflected upon the solutions, searched for a means to confirm sameness, and concluded the work was accurate by use of comparison and analogy. In fact, the paper had been folded to place one circle atop the other, held to the light, and the "pieces" analyzed to be identical size.

It seems significant that the problem solver recognized that the arithmetic process needed was division (e.g., "One pizza divided by 3"). By using speech to direct his action, he wrote 1/3 (i.e., "This, 1, and /, 3 over here [1/3]") but was unable to recognize thirds (e.g., "1 divided by 3 is 2"). Having reflected upon the symbol 1/3, the problem solver concluded that he did not really know the name. Apparently the problem seemed to be that the child did not know the language label necessary to clearly identify his solution although he did demonstrate considerable understanding of the concept, partitioning in thirds.

Summary: Language and Strategies Used When Solving Mathematical Problem Two. In summary, the specific steps taken to solve the mathematical problem and understandings achieved as revealed by John's action and language use were:

1. The child recognized that two problems existed within the task and that two answers were necessary. Therefore, the problem for him seemed two-fold: (a) to share the pizza among the boys and (b) to share the pizza among the girls. This recognition was not as evident when solving pizza problem one; seemingly, a transfer of learning had occurred.

2. Next John determined the answer for (1), using the halving mechanism by means of a horizontal cut to achieve sixths, as shown in Figure 4.1. Then the solution was revised by erasing to achieve thirds.

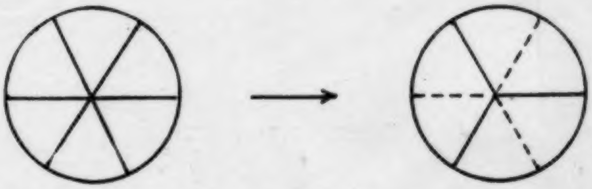


Figure 4.1

The Strategy Employed to Achieve Thirds

3. The solution for (2), (i.e., to share the pizza among the girls), was obtained by grouping and then partitioning in thirds to achieve one to one correspondence within the group. The stated reason for grouping was, "So each girl gets one piece of pizza so it will be even with the boys" rather than an analysis of details inherent in the problem (i.e., nine girls and three pizzas). It is interesting to note that partitioning in thirds no longer began with the halving mechanism. Apparently the concept of partitioning in thirds had evolved as shown in Figure 4.2.

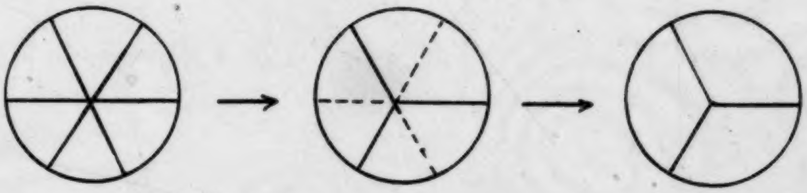


Figure 4.2

The Progression of Partitioning in Thirds

Pizza Problem Three

Given the task of sharing two pizzas among three boys and eight pizzas among twelve girls, John demonstrated that the following specific strategies were necessary.

I. Understanding the Problem. To clarify the problem, the problem solver seemingly assumed the questions were the same as those in the previous tasks as the questions were ignored and John first looked at the pictures. After focusing upon the pictures, he responded to the researcher's comment, "This one is a little trickier," by observing and analyzing as if to reason why the problem was trickier (e.g., "Cause this girl doesn't have a pizza"). When reflecting upon the meaning of the experience, reference was made to needs (i.e., "I don't want that one [girl]. This girl gets one whole pizza and the rest only get half").

II. Planning the Solution. By thinking aloud to himself, the problem solver explored several alternative solutions which were embedded in a series of reports and predictions. There appeared to be an effort to analyze and extract the significant meaning (e.g., "There's three for one here and there's three for two here") and to reflect upon one's own feelings (e.g., "This one is trickier," "might be easier if -- this is really tough"). The strategies within the predictive function focused upon recognition of possible alternative courses of action and prediction of the consequences (e.g., "Three boys. Let's see -- I put pizza in halves and a half left," "One pizza at a time, have to cut it into -- pieces --," "Give it [the second pizza] to the girls," "I think I'll try that. That's two [pizzas] each [group]"). Through use of prediction, the child had logically reasoned that both partitioning and grouping were necessary to solve the first part of the problem, that is, to share the pizza among the boys.

To predict the likely procedure necessary to solve the second part of the problem (i.e., to share the pizza among the girls), the child stated "Probably the same." Predicted consequences of the action were made by reference to detail while thinking aloud (e.g., "She's for that pizza. She's for that pizza. She's for that pizza. That would be -- two girls for another pizza -- think it is one pizza -- eight pizzas -- four more pizzas"). When the possible solution was not successful, the problem solver expressed his frustration and need for assistance (e.g., "I don't know how -- where's that teacher"). The utterance served a self-maintaining purpose.

When reviewing the problem solving process with the researcher, the child's statement, "three girls, one pizza," implies grouping had occurred. However, four pizzas remained. Through a series of questions, the thinking was clarified as shown in the following teacher questions and child responses.

"How many pizzas left over?"	"Four"
"How many groups of girls do you have?"	"Four"
"So what could you do with the four pizzas that are left over?"	"One more pizza for each group"
"-- So? --"	"How am I going to divide it?"

Language seems to have served two functions during the review outlined above—reporting and clarifying. By raising questions as a possible means of gaining understanding, the problem solver was enabled to proceed directly to the solution.

III. Solving the Problem. Only one utterance was expressed (e.g., "Ooops"). This monitored the action and directed new action (i.e., erasing) that resulted in the solution (i.e., thirds).

IV. Reviewing the Problem and the Solution. Language appeared to serve two important functions in the review of the solution—reporting and clarifying. By raising a question as a possible means of gaining meaning, the problem solver sought clarification (e.g., "How much did I get out of each pizza?"). Embedded in the report is the generalization that fair shares is the same amount (e.g., "Cause if it's fair shares, fair shares means you get the same amount").

Summary: Language and Strategies Used in Solving Mathematical Problem Three. As reflected in the talk, the solution to the problem was sought in the following sequential steps.

Part (1): (the boys)

1. "I cut this pizza into three pieces. Cause there was three boys"
2. "I cut up three more pieces"
3. "I gave each boy two pieces"

Part (2): (the girls)

1. "I gave each group of girls one pizza"
2. "I gave the leftovers to the groups. Each group gets two pieces"

3. "I divided the next pizza up into three so each girl gets two pieces"

Part (3): (Comparing)

1. "I took my fingernail and measured it," provides evidence that the child had sought to prove the accuracy of his solution, "they got the same."

Pizza Problem Four

Given the task of sharing four pizzas among seven boys and two pizzas among four girls, John employed the following strategies.

I. Understanding the Problem. The problem solver looked at the picture, reported his observation by referring to detail (e.g., "Oh! Oh! Now look at all the boys"), and, seemingly assumed that the questions were the same as those in previous problems as there was no rereading.

II. Planning to Solve the Problem. In a series of reports to the self, as revealed in the following examples, the language of the problem solver suggests serious efforts were made to reflect upon the meaning of the problem.

	Example	Strategy Used
1.	"-- some pretty tough"	reflecting upon the experience and his own feelings
2.	"The girls are easy to divide. I know that."	self maintaining
3.	"It wouldn't work. Two boys got one pizza cause there will be one left over."	anticipating a possible solution and consequence
4.	"What would I do? -- three pieces -- no, I have to --"	raises a question to focus and clarify the procedure

5.	"It can't be. One guy gets one whole pizza and one guy gets an itsy bitsy piece."	analyzes a possible solution; finds it inappropriate
6.	"It would be a little hard cutting it into seven pieces."	recognizes an alternative course of action and reflects upon the difficulty

By means of a question, "What did you say?" the researcher attempted to focus the child upon his last statement (see number 6 above). Ignoring the researcher, however, John focused upon the second pizza and predicted the same difficulty (e.g., "It would be a little hard cutting it [the second pizza] into seven pieces again"). He then directed his action and predicted success (e.g., "-- turn it around -- sure, I could do it").

III. Solving the Problem. Four utterances were made while solving the problem. The first monitored the activity (e.g., "four -- half"): the second predicted a better procedure (e.g., "I know I'll cut it into five [sic] pieces"): the third monitored the action (e.g., "1, 2, 3, 4, 5, 6, 7,"): and the fourth reported a successful solution (e.g., "I've got seven pieces"). Upon completion of the task the child sought by means of a question to clarify the solution (e.g., "What's my answer"), seemingly searching for a label.

IV. Reviewing the Solution. It appears that the problem solver used language for three purposes; namely, to clarify the meaning, to report the solution and to explain a process. To clarify the meaning, each of the original questions was read aloud and then the solution reported. To explain the process which enabled the child to determine which was more 4/7 or 1/2, the child appeared to have reflected upon a possible proof (e.g., "Let's see"), and reported that he had used the diagrams as the basis for proof. In fact, he had placed 4/7 atop 1/2 and recognized 4/7 was more. This implies the child had, by logical reasoning, sought a means to know which was more, lacking only the labels needed to convey the meaning.

Summary: Language and Strategies Used When Solving Mathematical Problem Four.

1. When given the task, the problem was recognized to require two answers.

2. The solution, one-half, was immediately recognized to part (2) of the problem and this part was answered first.

3. Once the answer was determined for part (2) (i.e., the girls), the problem solver endeavored to use the same process (halving mechanism) to achieve the solution to part (1) (i.e., the boys).

4. The process was initiated by grouping, which was unsuccessful. Then one to one correspondence was sought, using trial and error. In summary, the process evolved as shown in Figure 4.3. The solution was correct and recognized as "four out of seven." Again, no label was available. The size was reported as "pieces." Portions less than one-half appear to be labelled as "pieces."

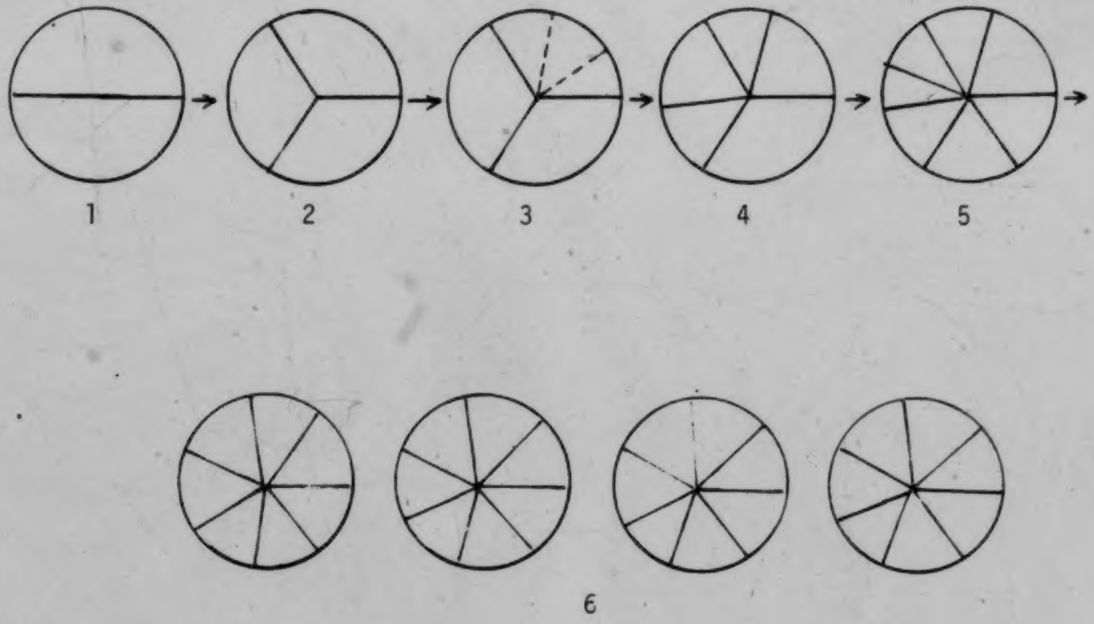


Figure 4.3

Evolution of the Procedure to Achieve Four-Sevenths

Pizza Problem Five

When presented with the task of sharing six pizzas among four boys and nine pizzas among six girls, John demonstrated that the following specific strategies were necessary.

I. Understanding the Problem. John immediately referred to the picture, focusing his attention upon the boys. Seemingly, the child assumed that there were two questions to be answered and that two solutions would be required.

II. Planning the Solution. The solution to the problem was planned in the following sequential steps.

Part (1): (the boys)

The linguistic activity of the child reveals an attempt to predict possible solutions and suggests strategies of logical reasoning to arrive at a conclusion as revealed in the following example:

> Two pizzas for two people and there would be two left over. So that might be three pizzas for two kids equals three pizzas for two kids. So I think they have to go that way.

The problem solver had referred to the detail of the problem, recognized possible solutions, reflected on the solutions and derived a plan. The use of "so" indicates awareness of relatedness and connectedness. "Might be" suggests that the child was aware of other possible solutions and "I think" precedes the conclusion.

Part (2): (the girls)

The functions of the linguistic activity while planning the second solution were to reflect on the meaning of the experience, including the child's own feelings (e.g., "That's a little tougher"), to predict a possible solution (e.g., "It has to be two for three"), and to clarify the process by questioning (e.g., "How can I divide into four pieces?"). At this point, John apparently equated two pizzas with two girls, became confused, referred back to part (1): (the boys) and re-directed his thinking and activity (e.g., " -- get three pieces -- so half right in the middle --"). The child had apparently analyzed, selected and discriminated. Of particular interest is the prediction made "It's two for three. It has to be two for three" since two boys share three pizzas in part (1).

III. Solving the Problem. After planning the solution no language was used to solve the problem of sharing the pizza among the boys.

To solve the problem of sharing the pizza among the girls, the activity of the problem solver was monitored, a strategy of the directive function (e.g., "One, two -- there's two for three, two for three, two for three") and the solution then reported (e.g., "Each girl gets three pieces -- three pieces").

IV. Reviewing the Problem and the Solution. In response to the question of the researcher, "How much pizza did each boy get?" the problem solver sequentially read aloud each of the printed questions indicating the need to clarify the questions, probably because this need had not earlier been satisfied. The child then concluded, "If they get fair shares, they would have the same amount." This statement and the attempt to determine the solution to sharing the pizza among the girls by using the same procedure which was successful to solve sharing the pizza among the boys suggests the problem solver may have projected the above conclusion into the second part of the posed problem. By 'talking it through' John endeavored to logically reason the solution. Twice he sought to clarify the question, "How much pizza does each boy get." First, a question was directed to the researcher (e.g., "In a group?"). Secondly, a question was directed to the self as if to focus attention (e.g., "How much does it go together"). John then reported, "If you put them together that would make one pizza and a half." The solution to sharing the pizza among the girls was reported and explained: "A pizza and a half. The two halves together and you would have one more half left over so it's three pieces. It's a pizza and a half." Through use of logical reasoning the child recognized the mathematical principle that three halves equal one and one half.

To describe the process, John reported:

> Okay, I figured it because, first I tried two people for -- two boys for two pizzas. That didn't work. So I tried three boys for -- two boys for three pizzas and they had -- I divided so -- so that each boy gets the same amount. Then girls by three pizzas for two girls and then I drew in a line to divide the pizza in -- pieces. That's how I figured it out.

The report indicates a planned solution and that the plan necessitated grouping. The arithmetic process was recognized as division and there was the need to achieve equal portions or sameness (e.g., "So each boy gets the same amount").

Summary: Language and Strategies Used When Solving Mathematical Problem Five.

1. When presented with the task, the problem was recognized to require two answers.

2. The solution for sharing the pizza among the boys was determined first. The procedure was to group (e.g., "equals three pizzas for two kids"), and then to partition in halves using horizontal cuts as shown in Figure 4.4.

3. The problem solver used the same procedure to share the pizza among the girls as had been successful to solve the problem of sharing the pizza among the boys (e.g., "Its two for three. It has to be two for three").

4. The problem solver concluded three halves is the same as one and one half (e.g., "If you put them together that would make one pizza and a half"). After attempting one to one correspondence, the process to achieve the solution evolved as follows:

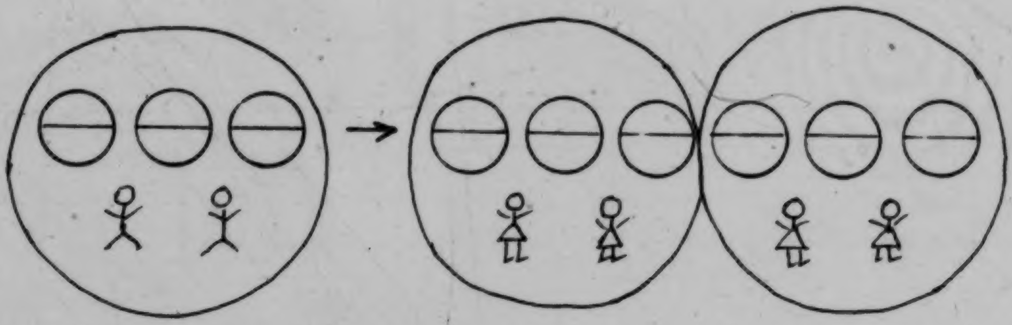


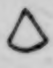
Figure 4.4

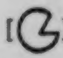
Transfer of Problem Solving Process to Achieve
One and One Half

Pizza Problem Six

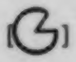
Given the task of sharing three pizzas among seven boys and sharing one pizza among three girls, John demonstrated that the following specific strategies were necessary.

I. Understanding the Problem. Ignoring the questions, the problem solver referred immediately to the picture and concluded, "The girls aren't hard," indicating a recognized solution and a reflection of his own feelings. Attention was then directed to Part (1): (the boys), (e.g., "Let's see if I can figure out the boys").

II. Planning the Solution. The language used by the problem solver as he monitored his own activities suggests that by a process of elimination, based upon prediction and reasoning, various possible solutions were reflected upon and discarded as inappropriate (e.g., "Let's see. One for one pizza. Nope that didn't work. Two for one pizza. No. Three for one pizza. Oh. Oh. I took my name -- [inaudible]"). Embedded in the process is a reference to the detail (e.g., "seven boys for three pizzas") which seemed to give rise to a new possible solution (e.g., "give one pizza to each boy, there would be four boys left over"). In response to the researcher's probe (e.g., "What happens if you try doing it the same way as the girls") the problem solver referred to the pictured solution, Part (2): (the girls), predicted (e.g., "I think I know what would happen"), monitored his activity (e.g., "If I divided this way -- these guys get it -- three -- these guys would get this. Then this guy would get that --") and concluded (e.g., "This one doesn't get near fair. He's not having a fair share"). The conclusion was supported by logical reasoning using comparison (e.g., "Cause these guys get a small piece and he gets a great big pizza"). This process seemed to provoke new insight (e.g., "Cut it in seven pieces!"). The researcher, by means of a question, tried to focus the child's attention (e.g., "What should he have?"). The child stated, "What he should have is something like that" and drew . This also apparently led to a possible new solution (e.g., "I know. Get seven more pieces").

III. Solving the Problem. The subsequent attempt to enact the predicted solution was accompanied by speech which monitored the action (e.g., "-- cut it [] into seven pieces but then it won't be fair shares. Let's see. Umm"). By partitioning in thirds, the child neared completion of the solution. The question, "What are you going to do with the left over part," was responded to by, "I don't know. [pause] Cut it into seven more pieces!" The activity was monitored, (e.g., "1, 2, 3, 4, 5, 6, 7"), corrected, accompanied by monitoring (e.g., "More pieces have to go. Not big enough") and reported (e.g., "I divided the one piece left over").

IV. Reviewing the Problem and the Solution. In response to the questions of the researcher, the child reported his answer and sought to clarify the researcher's questions by means of his own questions, as evidenced in the following dialogue:

Researcher	Child
"Can you tell me how much each boy gets?"	"One piece."
"How big is that piece?"	"How big is that piece?"
"One piece out of how many?"	"Three."
"How much does each girl get?"	"One."
"Out of how many?"	"Out of three."
"What did you divide that [] into?"	"Seven pieces."
"Who did you say got more?"	"The boys."

When explaining the process reference was made to detail and again justified the reasoning (e.g., "There was seven boys here and each one of them got one piece and there still was some more left over so I decided to cut that in seven's. -- One more time, so they each got two pieces"). Labels were not available to report the size and reference was not made to the concept inherent in the problem, that is, three pizzas, seven boys means three-sevenths. The overriding notion seemed to be that fairness meant all the children should receive the same amount.

Summary: Language and Strategies Used when Solving Mathematical Problem Six.

1. When presented with the task, the problem solver recognized the problem required two answers.

2. Part (2) (the girls): The problem solver stated, "I know. The girls aren't hard" and immediately determined the solution by partitioning in thirds.

3. Part (1) (the boys): The problem solver while monitoring

the activity sought to determine the solution by means of trial and error. He then resorted to applying the solution to (2) sharing the pizza among the girls, that is, partitioning in thirds. John then partitioned "the part left over" in sevenths by using radial strokes. In summary, the solution evolved as shown in Figure 4.5.

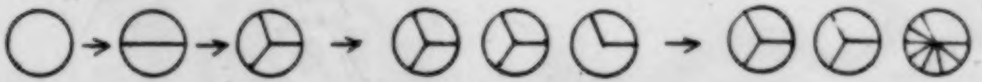


Figure 4.5

Five Progressive Steps to Share Three Pizzas
among Seven Boys

Problem Solving Strategies of the Small Group: Grade Two

For each of the four pizza problems the major strategies demonstrated by the group will be reported.

Pizza Problem One

The three children solved the first problem as follows:

I. Understanding the Problem. The initial activity of one child was to embellish the pictures of the pizza. In response to the question of the researcher, "Why are you doing that?", the child stated, "I'm putting on the pepperoni," and then justified the activity, "'Cause these pizzas only have cheese." The child obviously realized the pizzas were not real but seemingly elaborated the problem through use of the imagination based on real life experience and was directed from the activity by a second child (e.g., "Do that later").

Endeavoring to clarify the problem the problem solvers initiated two procedures. First each child read aloud to the self the five posed questions (stated in Chapter 3). Secondly the same questions were reread aloud to each other. Meaning was seemingly brought to the problem by use of two strategies: reading aloud to the self and reading aloud to others.

II. Planning the Solution. Reference to the pictures and noting the detail was made by Karla when planning the solution (e.g., "Look it. 1, 2, 3, 4, 5, 6, 7. There's seven pizzas and there's 1, 2, 3, 4, 5, 6, 7—7 kids"). By counting aloud, Helen confirmed the detail (e.g., "1, 2, 3, 4, 5, 6, 7. 1, 2, 3, 4, 5, 6, 7. Yeah") and Karla concluded, "So they each get one." Within the directing function of language, two strategies are apparent: directing the action of others and monitoring the actions. It is of interest to note the two problems (how much pizza for boys and how much pizza for girls) were not separated but treated as one problem (pizza for "kids"). All reached a consensus of opinion (e.g., "Yeah. They each get one").

III. Solving the Problem. By means of a question, effort was made to clarify the desired procedures ("Do we put a ring around them") and followed by the use of language to direct the activity (e.g., "Put a ring around"). To the researcher's question, "What are you doing now?" Helen reported, "I'm putting it in his hand so he eats it nicely." Imagining, based upon real life experience, seemed to influence the group activity as all then drew the pizza in the hand (e.g., Karla: "Yeah. Cut it in pieces—the same kind of pieces") and with further direction (e.g., Annie: "Make sure they're the same. Cut them right"). Implied is the notion that all pieces should be the same size but each group member need not do the same (e.g., Annie: "Your's are different than mind." Karla: "Doesn't matter long as they're the same"). This rather confusing response seemed to be intuitively understood to mean as long as your pieces are all the same, and my pieces are all the same, then it doesn't matter that yours are different from mine, i.e., all the same size.

IV. Reviewing the Problem and the Solution. To review the solution the children first used language to direct the activity (e.g., "Let's see. Now let's do the questions"), then to clarify the questions each member of the group read the questions aloud to the self and then used language to determine who was to report the solution (e.g., "Who writes the answer?"). Karla replied with justification (e.g., "We all do. She gave us each a page"). The group agreed and recorded each boy got one pizza, each girl got one pizza, and this was the same amount but the stated reason was given by Annie, "Cause it's fair shares." To explain the process the group reported they had referred to the pictures ". . . we counted the pizzas and we counted the kids, and we put a ring around." Partitioning, though done by all, was not reported.

Throughout the task, the language reveals relational strategies which recognize others (i.e., acknowledging, agreement, seeking consensus of opinion).

The relationship among the three children is inferred from the way each approaches and responds to the other. An assumed equality of friendly mutual support prevails as the basis for their talk during the assigned tasks and through to achieving an agreed solution. At times the relationship seems under strain as the leadership role becomes dominant and accuracy and approval are sought. Once positive recognition of achievement is affirmed the group automatically proceeds harmoniously and renegotiation of the leadership seems to be unnecessary.

Summary: Language and Strategies Used When Solving Mathematical Problem One.

1. The problem was not recognized to have two parts, but treated as one problem.

2. One to one correspondence was made relating one child to one pizza.

3. Three types of partitioning occurred (halves, quarters, eighths). All began with the halving mechanism using a horizontal cut. The pizza was then drawn "in the boys' hands."

Pizza Problem Two

I. Understanding the Problem. First, the children looked at the pictures. When the problem was surmised to be harder (e.g., Karla: "This is harder"), it was recognized and acknowledged by Annie who, while monitoring her own activity, reassured the group by disagreeing and predicting the solution would be easy (e.g., "No! This is going to be easy"). Secondly, the children sought to clarify the problem by use of two strategies within the clarifying function: each read the five posed questions aloud to the self and then simultaneously reread the same question in sequential order to each other. No comments were made about the questions during this process.

II. Planning to Solve the Problem. Having read the questions Karla initiated the plan (e.g., "Well, you'd better do the boys first"). Acknowledgement and agreement were expressed (e.g., Annie: "Okay") and the possible solution predicted (e.g., Karla: "Three. No"), synthesized (e.g., Karla and Annie: "One piece each") and group consensus resulted (e.g., "Three"). Annie announced to the group "And I know how to make it too." Acknowledging and responding to the comment by means of a report, Karla remarked "I don't." The use of language while co-operatively planning the solution was to direct, predict, synthesize and report. The relational strategies apparent were agreement, acknowledgement and consensus.

III. Solving the Problem. Each part of the two-fold problem was discussed as follows:

Part A (the boys): Having agreed in the plan that three pieces were required, each child independently attempted to achieve the solution. Annie, while monitoring her activity, immediately partitioned in thirds (e.g., "You go like that, like that, like that. Hmm") and reported achievement (e.g., "There. Like that. I made it"). Group members sought assistance (e.g., "How did you do it?")~~,~~ and a leadership role of evaluator and instructor emerged. Karla presented her work (e.g., "Just lookit") and the response of Annie was to evaluate (e.g., "Those aren't even"), project personal feelings about the problem, perhaps empathizing with the other child (e.g., "It's tricky") and report past experience with this type of problem (e.g., "I did it once when we were playing on the board"). Karla again sought approval and, as evaluator, Annie assessed the work (e.g., "It's still not even, Karla"). An attempt to defend herself and justify the work was apparent (e.g., "Look, one-two-three. There's three there"). These strategies are within the maintaining function of language. The response of the evaluator expressed agreement and by use of logical reasoning, Annie justified the evaluation (e.g., "Yes, but they're not fair"). This was repeated to Helen (e.g., "There's three but they're not fair shares"). The leadership role became that of instructor and demonstrator (e.g., "Here's the pizza and there's the middle and so you go [shows how to divide the pizza into three fair shares] like that, like that and that"). Both Helen and Karla again attempted to complete the task and to have the work approved (e.g., "Is that better, Annie?") and to be met with the response, "No."

Apparently discouraged with her progress, Helen reported an alternate possible solution (e.g., "I know another way of making three. See? Like this. You can make a "T" like this. See? One-two-three"). Annie again evaluated and justified the evaluation (e.g., "There's three but --"). As evidenced below a serious effort was made to enable all to achieve three fair shares.

	Language—Examples	Language—Thinking Strategies
Karla:	Let's see yours.	- directing to receive guidance.
Annie:	Okay, here's the pizza and here's the middle. You go there, there, there. [Demonstration]	- a report accompanied by explanation and directing the procedure.

Karla:	Oh. I see.	- acknowledgement, an effort to maintain the self, and seemingly impetus to try again.
Karla:	There, is that better, Annie?	- seeking approval by means of a question.
Annie:	No.	- reports evaluation.
Helen:	I know another way of making three. See? Like this. You can make a "T" like this. See? One, two, three.	- reports a possible alternative. Seeks approval twice by means of questions. - justifies the solution.
Annie:	A "T"? Okay, now here's the circle -- [Demonstration]	- questions the use of a "T". Begins again to explain how to arrive at the solution. - directs the activity.
Helen:	One-two-three-four. Hey, that's not fair. That's not fair.	- analyzes own work. - realizes the error. - justifies the realization.
Annie:	Because they're not fair, they'll fight.	- justifies the conclusion using an imaginative projection based upon real life experience.

The functions of the language used in the above passage are to clarify, evaluate, report, direct, logically reason and imagine. The strategies within the functions are questioning, predicting, collaborating to direct an action, and justifying. The relational strategies are mutually supportive, recognizing and acknowledging needs of others, co-operation, and agreement as well as those which are self-assertive.

Part B (the girls): In response to the urging of the researcher, the group proceeded to the second part of the problem. Before attempting to solve the problem the group apparently recognized the need to plan a new solution. This procedure was initiated in two steps: (1) each child read aloud all posed questions to the self;

(2) each child simultaneously reread the questions to the others. This seemingly clarified the problem. Using speech to monitor the action, the details inherent in the problem were noted (e.g., Helen: "There's 1, 2, 3, 4, 5, 6, 7, 8, 9 girls"), confirmed (e.g., Annie: "There are nine girls"), and restated (e.g., Helen, "Ten [sic] girls. And there's only three pizzas"). The conclusion (e.g., Karla: "Three girls will share one pizza and three girls will share one pizza and the other three girls will share the other pizza") was reported and agreed upon (e.g., "Yeah").

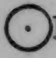
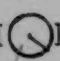
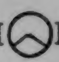
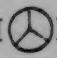
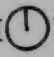
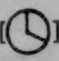
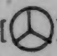
Annie apparently noticed the error, "Ten girls," then confirmed and reported the count (e.g., "Nine—there's nine kids"). Karla perceived that a mathematical fact, 3 + 3 + 3 = 9, may relate to the solution (e.g., "3 + 3 + 3 equals 9") and Helen concluded (e.g., "So these guys can share three and these guys can share three and --"). The implication seems to be "three guys" share three pieces. Annie supported the conclusion as she monitored a planned activity while referring to the diagram (e.g., "Okay, so there's one-two-three; one-two-three; one-two-three. That makes nine").

As the solution was being sought, although all knew each pizza was to be divided into three, only Annie was able to proceed directly to the solution but not without difficulty:

> Annie: One-two-three. One-two-three, oh, oh. I didn't make it right this time. Where's the eraser? Here it is. Now let me check how I did it last time. There's one going down like that to the middle and one going like that and one going like that.

Language was used to monitor the activity, to focus attention, and to analyze the exact procedure necessary. Realizing her work to be incorrect, Karla sought further assistance from Annie (e.g., "There I did it. I did it! No good! Annie, how did you do that? Will you do it for me?"). The response (e.g., "No way") was emphatic. Only the intervention of the researcher perhaps prevented a quarrel (e.g., "Why not show them how to do it? Show them carefully how to do it"). Helen, perhaps aware of an emerging conflict attempted to set the relationship on a better footing by appealing to reason (e.g., "That's sharing and that's co-operation"). 'Sharing' was articulated to mean an integral part of co-operation, a high level of conceptualization. The concept of co-operation by inverse analogy is seen to include 'sharing' as helping or assisting one's peers. This restored the equality and once more a mutually supportive relationship emerged. Again the procedure was carefully explained by Annie, this time accompanied by demonstration:

> Annie: You make a circle here, and here's the circle. Look at the circle, there. Now there's the pizza and here's the middle and then take the middle of the pizza and go like that and that one goes like that and that one goes like that.

For the first time, Karla achieved an accurate solution meeting the approval of Annie (e.g., "Yup. Yup. That's right"). Transfer of this skill was difficult and later Karla again sought Annie's recognition (e.g., "I did it, look"). However, the work was evaluated to be inaccurate and the error pointed out (e.g., "No. Look! Still -- look. See?") and Karla returned to the task, monitoring the activity as she worked (e.g., "There - there - there"). Again by questioning and showing the work Karla sought Annie's approval (e.g., "There, like this? I did it Annie. See Annie? See, look, one-two-three; one-two-three"). Annie again attempted to clarify the process by showing her work and explaining (e.g., "I'll show you one more sample. Now watch. See?"). Karla agreed and directed Annie to show her again ("Okay, show me") and Annie again explained the procedure using demonstration (e.g., "There's the middle [] and you go out [] and then like this [] and then straight down []"). Evaluation was again sought by Karla (e.g., "Is that fair?"). The partitioning was assessed, the assessment reported and the work corrected (e.g., Annie: "No, you did it wrong. Like that. There"). Karla returned to the task, affirmed her understanding, monitored her activity, and then sought Annie's approval (e.g., "Oh I know now. Hmm, hmm, hmm. You go up and then you go -- like this, Annie?"). Ignored by Annie, Karla returned to the task, expressed her feelings, monitored the action, and again sought Annie's approval (e.g., "I'm worried about this drawing. Hmm. I don't draw very well. That goes up [], that goes out [] and that goes like that []. Like that, Annie?"). The work was approved ("Yup"). Seemingly, after many persistent efforts, Karla realized the mechanics of the procedure.

As Annie and Karla were working together, Helen had monitored her activity, realized the work to be incorrect and at the same time observed Karla's struggle (e.g., Helen: "No, there's four. Can I use the eraser, Annie? Okay, Karla, now there, there's the middle --"). Evidenced is a relational activity recognizing ownership. Of interest is Helen's attempt to direct the activity of Karla though unable to do the same task. The response by Karla was to assert the self (e.g., "I KNOW how to do it"). The message was definitely not to interfere. Thus Helen assumed the role of observer and later as that of a self-appointed recorder.

IV. Reviewing the Solution. In response to the question, "How much pizza does each boy get?", the group reported, "One piece each" and then clarified the solution to mean "one piece out of one pizza." To enable the children to respond to the next question (e.g., "That's right. But also one piece out of how many pieces?"), it seemed necessary that the work be monitored (e.g., "One, two, three") before answering "One piece out of three." Helen, the recorder, presented a complete report:

> [Busy spelling "piece."] One piece - P - I - E - C - E each. Okay, I'll tell you what I wrote. Okay, how much pizza does each boy get? I'll tell ya. I wrote - "They each get one piece out of three pizzas." And then for girls I wrote - "Each girl gets one piece out of three pizzas."

From this report there arose debate, clarifiction of the answer, and agreement:

> Helen: There's three pizzas and each girl gets one from out of three pizzas.
>
> Annie: One pizza.
>
> Karla: Each girl gets one out of one pizza. Each girl gets one out of one whole pizza.
>
> Helen: BUT there's three pizzas.
>
> Karla: One group of three gets one, one group of three gets one and one group of three gets the other one.
>
> Helen: Yeah, each girl gets one piece out of one pizza.

No labels were available.

The question "If each child gets a fair share, who gets more, each boy or each girl" was clarified individually as each child read the question aloud and then the question was reread in response to Annie's directive (e.g., "Karla, read the question again very loud to everyone"). The answer of all (e.g., "They get the same amount") was justified through use of logical reason.

> Karla: Because there are three pizzas for the girls. One group of girls would get one - one group of three girls would get one pizza, one group of three would get the other and the other group of three would get the other.

The process to solve Part A (the boys) was elaborated as referring to detail.

> Karla: The boys -- well, we split it in three pieces so one boy can have one slice, the other boy can have the other slice and the other boy can have the other slice.

Apparently "slice" and "piece" have equivalent meaning. The group then made an attempt to explain how the problem had been solved:

> Karla: Because we counted the pizzas and we counted the things together.
>
> Helen: And we counted them all and see, they all get the same. The girls get as much pizza as the others.

Summary: Language and Strategies Used in Solving Mathematical Problem Two. In summary, the specific steps taken to solve the mathematical problem and understanding achieved as revealed by the interaction and language use were:

1. The group recognized two problems existed within the task and that it was necessary to determine first how much pizza each boy gets, Part A (the boys), and then to determine how much pizza each girl gets, Part B (the girls).

2. Part A (the boys): The group plan was to divide the pizza

into three pieces. The solution was achieved as shown in Figure 4.6.

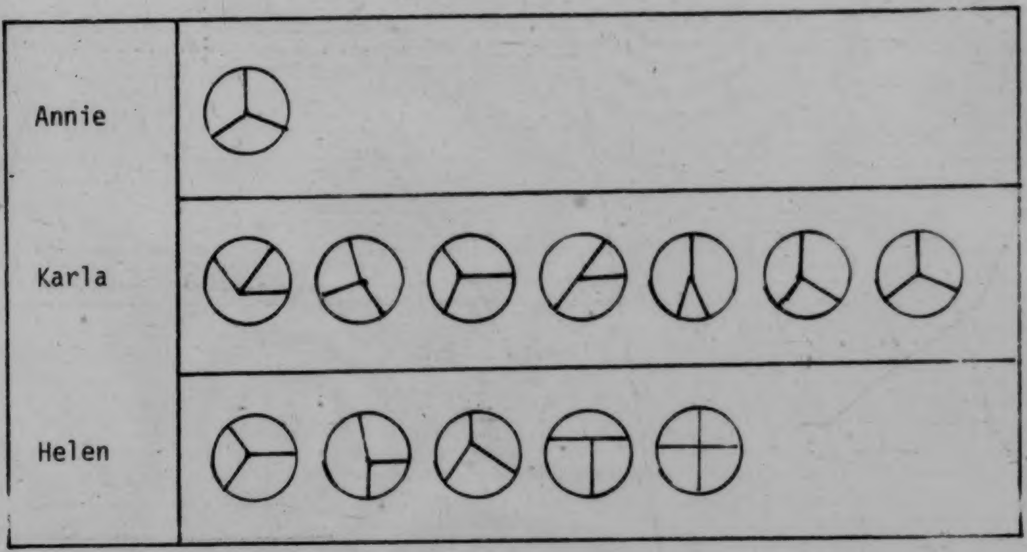


Figure 4.6

Partitioning Behaviors in Thirds

Annie immediately partitioned in thirds. Her mother later confirmed a blackboard was used frequently at home while playing with an older brother but the child had not been taught at home to partition in thirds. Karla struggled to determine the middle and then rotated three radii. Helen could not find the middle and sought the solution using analogy to a "T" then analogy to a "†". She never completed the task.

3. Part B (the girls): The solution was obtained through use of the following procedures: (a) grouping and (b) partitioning in thirds to achieve one to one correspondence. The reasons stated for grouping made reference to detail (i.e., "There are nine girls and only three pizzas"), used an additional fact (e.g., "3 + 3 + 3 = 9"),

and was explained (e.g., "One group of three girls would get one pizza, one group of three would get the other and the other group of three would get the other"). Partitioning in thirds apparently arose from the idea of three girls sharing one pizza.

Pizza Problem Three

The problem required that the children share two pizzas among three boys and eight pizzas among twelve girls. Strategies for solving the two-part problem (Parts A and B) will be discussed separately.

I. Understanding the Problem—Part A (The Boys). To understand the problem, three procedures were initiated by the problem solvers. First, each child simultaneously read all questions aloud to the self. Secondly, all questions were read aloud to each other. Thirdly, reference was made to the detail of the picture (e.g., Karla: "Hmm. Cause since there's two pizzas and we have three boys --").

II. Planning the Solution—Part A (The Boys). A possible solution suggested (e.g., "Helen: "They each get three. They each get three pieces") was examined using partitioning and then rejected by the group. The proposal and activity enabled Annie to perceive the correct solution (e.g., "Oh! I see. Each boy gets two pieces") which in turn seemingly enabled Helen to clarify and restate her meaning as she monitored the partitioned diagram (e.g., "Yeah, that's what I meant. See? One-two, one-two for him; one-two for him; one-two for him"). Karla sought concurrence (e.g., "Do you agree on that") and the group expressed consensus (e.g., "Yeah"). It is of interest to note that Helen is able to recognize partitioning in thirds, justify the meaning of the solution but again is unable to partition in thirds. Only Annie diagrammed the solution. Monitoring the spelling, Helen recorded the answer (e.g., "T-W-O") but was directed by Karla to use the digit (e.g., "Put the number"). Group effort made achievement of a solution possible.

The group proceeded to plan the next solution, Part B (the girls).

I. Understanding the Problem—Part B (The Girls): All monitored counting the detail of the problem, 12 girls. A relationship in the detail was recognized, concluded and reported (e.g., Annie: "So there's two pizzas for each group of girls"). The question was again reread aloud by Karla as if to clarify, then the number of girls was monitored in groups of three to the self. Karla then directed the group (e.g., "Everybody just stop and listen a minute"), and attempted to explain how the answer could be derived but she could not complete the explanation (e.g., "It says how much pizza does EACH girl get? We're supposed to count the girls and then count the amount of pizzas and then if there are eight girls, say, okay, so --"). The statement "We're supposed to count the girls" seemed the impetus to redirect Helen to the problem for she again counted the detail using her language to monitor the activity. Karla affirmed the count (e.g., "There's 12 girls"), was responded to by questioning (e.g., "There is?") and again all monitored the activity as they individually recounted. The question, "There is?", seems to indicate that the conclusion "there's two pizzas for each group of girls" may have been based upon this relationship being perceived and used to successfully achieve the solution to Part A (the boys).

II. Planning the Solution: Part B (The Girls). To enable the group to plan the solution, three times Karla endeavored to elaborate and re-present her understanding.

	Language—Examples	Language—Thinking Strategies
Karla:	Twelve girls and there are only 1-2-3-4-5-6-7-8—and there are eight pizzas. Four plus four equals eight - each girls would get two pieces! Each girl would get one piece each! Who agrees on that?	Refers to detail, recognizes and reports a possible related mathematical fact, suggests a solution, seeks agreement.
Karla:	Okay, there are eight pizzas and twelve girls. And four and four is eight. Four plus four is eight. You guys, four and four is eight. There are 1-2-3-4 ... 8. This is a hardy!	Refers to detail, reports and directs attention to a possible relevant additional fact, monitors activity, reflects upon the experience.
Karla:	Four plus four is eight. Four plus four is eight, so 1-2-3-4-5-6-7-8. There aren't enough pizzas. So that means we have to take away.	Repeats the relevant additional fact. Concludes.

Report of an alternate procedure by Annie (e.g., "I'm making them into groups. There are two pizzas for each group of three kids"), was extended using prediction and monitoring by Helen (e.g., "There will be 12 pieces left. [Begins to count.] 1-2-3; 4-5-6; 7-8-9; 10-11-12; 13 -- Oh"), then confirmed by Annie (e.g., "Yeah, that's right. I counted it").

III. Solving the Problem (Parts A and B). The solution was achieved in concert with the plan. The process was grouping followed by partitioning in thirds. Karla stated the solution (e.g., "Each girl gets one piece"), which was corrected by Helen who justified her correction ("No, each girl gets two pieces. Yeah. We figured it out") and the group expressed agreement.

IV. Reviewing the Solution. The answer (e.g., "Two pieces") was recorded for each of the first two questions. Karla, by reading aloud, focused attention of the group upon the next question (e.g., "If each child gets a fair share of the pizza, who would get more, each boy or each girl?"), and the answer agreed upon (e.g., "Each of them get the same amount").

As follows, the process was revealed by the group in a series of reports.

		Strategies	
Language—Examples		Language-Thinking	Math-Thinking
Annie:	I made them into boxes.	Reporting	Grouping
Helen:	We made them into boxes and then we figured it out—then we figured out how much each girl got.	Reporting Explaining a process	Grouping
Karla:	You had to put 't' in the centre.	Reporting Makes an analogy to label the process of partitioning in thirds	Partitioning
Annie:	I put the girls into boxes and the pizzas into boxes of two and then I cut the pizzas into pieces.	Reporting to explain a process	Grouping Partitioning
Karla:	She divided them in three. Two pieces each.	Reporting Concluding	Specifies the type of partitioning

When specifically addressed to explain her procedure (e.g., "And how did you figure it out, Karla?"), a reprimand (e.g., Helen: "You never drew it!") provoked the following response, including justification to maintain the self and an imagined though accurate explanation:

> Karla: I did it in my head! There are two here. There are only two pizzas and two -- well -- two pizzas and three girls so I divided them into pieces -- three -- pieces -- I had three pieces in each one. And then there are two in there and two in there and there wasn't enough left. So in one I had one more left and in the other I had one more left. So then I took them and I added them together and that made two so each girl got two. This one got two and this one got two and this one got one from there and one from there—that's two!

The child clearly envisioned the solution though unable to demonstrate accurate partitioning in thirds.

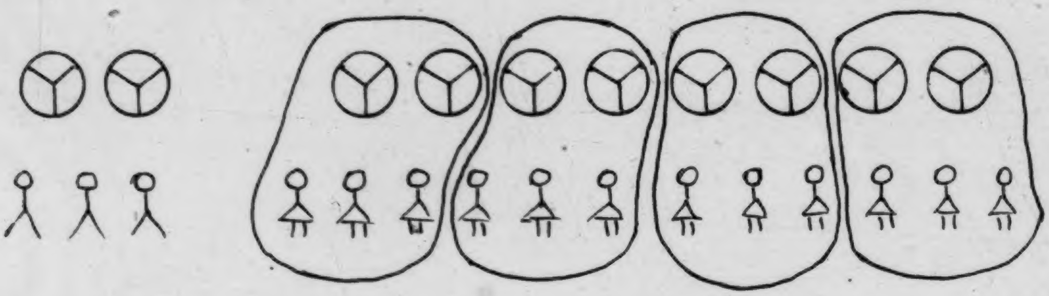
Summary: Language and Strategies Used to Solve Mathematical Problem Three.

1. The group clarified the question, then analyzed the picture to determine the answer to the first question.

2. The solution for the first question, Part A (the boys), was achieved by first partitioning in thirds then giving each boy "two pieces [2/3]" (see Figure 4.7). The process of distribution was explained by Karla as "two pieces for him, two pieces for him and this one and that one makes two pieces for him."

3. The solution to the second question, Part B (the girls), was achieved by first, grouping; secondly, partitioning in thirds; and lastly, giving each girl "two pieces [2/3]"), as shown in Figure 4.7.

4. It was concluded "each of them [each boy and each girl] get the same."



Part A
(The Boys)

Part B
(The Girls)

Figure 4.7

Partitioning in Thirds by Means of Grouping

Pizza Problem Four

Given the task of sharing four pizzas among seven boys and two pizzas among four girls, the group employed the following strategies.

I. Understanding the Problem. To obtain meaning that made solution of the problem possible the group seem to have taken the following steps:

1. An observation of the picture was reported (e.g., Helen: "There's more boys than girls this time").

2. Simultaneously, each child read all questions aloud to the self and then the same questions were read to each other.

3. By means of report and demonstration, an attempt was made by Karla to establish a procedure which would seemingly eliminate confusion for the group:

> This is what to do. I don't want to read all of them at the same time so I go like this. How much pizza does each boy get? [She covers up the other questions with paper.] And then I move it down and I read the next one. Be quiet -- the thing is on. How much pizza does each boy get?

4. The behavior required seems to be a synthesis, partly determined by the procedure established in the previous problem solving situations. And lastly, the attention of the group was directed to the first question (e.g., "Now the first one says, how much pizza does each boy get?").

II. Planning the Solution—Part A (The Boys). Several plans to determine the solution were explored, reported and rejected as revealed in the following samples.

	Language—Example	Strategy Used
Annie:	Four pizzas and I made a dot in the middle, okay. So this one would belong to these guys and this one would belong to these -- These two would belong to these three.	- (1) refers to detail and (2) reports a procedure (grouping) which reveals (3) an attempt to relate the two aspects of the detail in the problem, boys and pizza.
Helen:	Well, there's three here and there's three here and there's one guy left, Karla, and he eats a lot.	- reports a procedure which reveals attempts to group one aspect of detail inherent in the problem (boys). - imaginative use of language seeming to imply all detail must be included.
Helen:	No, no, no, no. Two for them, two for them, two for them. Each boy gets two. This boy gets a pizza and he will share with this guy, and this boy gets a pizza and he will share with this guy, and this boy gets a pizza and he will share with this guy, this boy gets a pizza and he doesn't have anybody to share with so what he will do is --?	- reports a procedure and analyzes a possible solution. - raises a question to complete the answer partitioning one pizza: thirds, one pizza: quarters.
Annie:	One-two-three. One-two-three four. That makes seven.	- monitors a possible solution. Reports achievement.
Helen:	Yeah, that's fair.	- recognizes other, expresses approval.
Annie:	If that's into four pieces?	- raises question to focus attention.
Helen:	There's three pieces here and four pieces there. One's left and Mum can put that in the freezer.	- (1) reports alternate solution; (2) grouping and partitioning in quarters; (3) completes solution by imaginative projection based upon real life experience.
Karla:	There's six men here - I've got a better idea. I have an idea. Okay. Since there are seven boys and there are only four pizzas, two boys can go	- suggests alternatives.

	together. Two boys can go together and this one --	
Helen:	And then there will be one left over.	- predicts
Karla:	Then these two boys can go together and these two can go together and this one can be left over, and then -- I've got it Annie!	- predicting a possible solution based upon halving.
Annie:	What are you going to do if there's pizza left over?	- raises a question to focus attention upon error. - seemed to clarify her own thinking and gains new insight.
Karla:	All these four boys can share out of these two and these three boys can share out of these two.	- reports a possible solution which is based upon grouping and not recognizing all the detail.
Annie:	I think we should take these pizzas and make them into three. We can go --	- predicts a possible solution which serves as impetus for the activity.
Helen:	We'll have two pizzas with four and two pizzas with three. Come here, I've got the answer. These guys can share that four, these guys can share that four and these guys can share this four. That solves it.	- returns to prior solution. - directs the group. - elaborates plan and concludes solution is appropriate.
Karla:	This boy gets one pizza and another boy gets half of a pizza and another boy gets?	- reports a solution which halves three of the pizzas, and leaves one whole pizza.
Annie:	Wait, wait a minute. That doesn't sound fair to me. One boy gets a pizza and the other boy only gets a half a pizza. That does not sound fair.	- directs the activity. States and justifies an opinion based upon logical reasoning. Reaffirms the opinion.
Karla:	We put a half to this boy and a half to this boy and then we get this pizza and this boy can eat just one half and one half will be left over.	- reporting the consequences of a possible solution.

II. Planning the Solution—Part B (The Girls). To plan the solution the group looked at the picture and reported (e.g., Annie: "This is simple"), referred to past experience (e.g., Karla: "Yeah. We did this in the other one"), and justified the report with explanation (e.g., Helen: "You just cut the two pizzas in half and these guys can have two [pieces] and these guys can have two [pieces]"). The justification indicates the process would be to group one pizza and two girls.

III. Solving the Problem. No speech accompanied the activity to solve the problem. The solution to Part A (the boys) was achieved by partitioning in halves using a horizontal cut. This solution was later modified in the review. The solution to Part B (the girls) was immediately recognized and achieved by partitioning in halves using a horizontal cut.

IV. Reviewing the Solution. First, the solution to Part B (the girls) was reviewed seemingly because it was the first to be achieved and recognized to be correct. The answer was reported, clarified, and agreement sought (e.g., Karla: "A half a pizza. Each girl gets a half a pizza. A half of one pizza. Do you agree?"). The question, "Do you agree," provoked a discussion using logical reasoning to state a mathematical principle: "two pieces [two quarters]" is the "same as 1/2"; as evidenced in the following examples:

	Language—Examples	Strategy Used
Annie:	There are four girls and there are two pizzas and we can split one pizza in half and then the other and then that would be fair. I split the pizzas into four and I had two groups and each girl gets two pieces.	- reports an alternate solution determined by grouping and partitioning in fourths.
Karla:	But that's the same as this Annie. When you split it in half it's still the same thing—you get two pieces and that's the same as one half.	- reports and illustrates by use of diagram two pieces is 1/2.

In reviewing the other half of the problem, Part B (the boys), the reported solution (e.g., Karla: "Each boy gets a half a pizza") was not accepted and agreed upon by the group (e.g., Annie: "A half, I think"). "I think" suggests tentativeness, an awareness that there may be a better solution. Justification for doubt and in effect to an effort to maintain the self was expressed (e.g., Annie: "I'm not the one who figured it out. Karla figured it out") and a realization stated by Annie (e.g., "Karla, we haven't figured out the boys yet"). The new solution (e.g., Helen: "Let's skip the boys out") was recognized and emphatically rejected (e.g., Annie: "No way!"). The response (e.g., Helen: "Okay, let's just read the next question") was acknowledged (e.g., Karla: "We can't do that, Helen, until we do the boys") and when the question was read despite the acknowledgement (e.g., Helen: "How did you figure this out?") the acknowledgement was expanded and explained (e.g., Karla: "That's what we are trying to find out").

Returning to the task of sharing four pizzas among seven boys, the group again used the halving mechanism and the following alternatives were posed to solve the problem of what to do with the remaining half:

"This boy can eat just 1/2 and 1/2 can be left over."

"They can share it."

"Split it in half."

"Split it in more pieces."

"No, each take a bite out of it."

The solution advanced from "left over" to sharing to halving to "pieces" to "bites." Nevertheless the group recognized that the girls' share of one-half was less than the boys' share of one half plus a bite:

> Helen: Yeah, each boy gets a half of the pizza and after they eat the half then they can each get a little bite of the other half. After they each have one half a pizza then they each get a little bite of the other half so the boys get it. The boys get more.

Summary: Language and Strategies Used When Solving Mathematical Problem Four.

1. When presented with the task, the problem was recognized as requiring two answers.

2. By a process of trial and error, the solution was sought but not achieved to Part A (the boys). Upon a return to the task, the solution was achieved by use of the halving mechanism and sharing the "left over half."

3. The solution to Part B (the girls) was immediately recognized and achieved by means of the halving mechanism, using a horizontal cut. One child partitioned in quarters but initiated the procedure using

the halving mechanism, horizontal cut. One half was recognized to be equivalent to two pieces, actually two quarters.

4. It was recognized that the boys got more pizza than the girls.

SOLVING THE PIZZA PROBLEMS: GRADE THREE

The problem solving strategies of an individual grade three child and a small group of grade three children will be specifically described in that order in the following discussion.

Problem Solving Strategies of the Individual Grade Three Child

Unlike the grade two child, Davey was able to proceed to solve the problems quite independently. Little direction was given when presenting the tasks nor was direction sought by the child while solving the problems. Thus the researcher was enabled to refrain from interacting with the child other than to probe to gain insight regarding the knowledge the child brought to the task and the behaviors which were manifested.

Pizza Problem One

Given the task of sharing two pizzas between two boys and five pizzas among five girls, Davey evidenced the following specific strategies.

I. Understanding the Problem. In contrast to the grade two child, the grade three individual child ignored the questions and, referring to the picture immediately, suggested knowing the answer (e.g., "I think I already know this") and proceeded to review the solution. Little or no planning seemed necessary.

II. Solving the Problem. No oral language was used while solving the problem. Neither partitioning nor an attempt to achieve one to one correspondence was evidenced.

III. Reviewing the Solution. In sequence each question was read and answered aloud by the child. The explanation to the question, "How did you figure this out?" indicates that unlike the grade two child, Davey found the solution easily and had recognized two questions were inherent in the provided illustration (e.g., "Easy cause there's two boys and there's five girls. There's five pizzas for the girls and there's two pizzas for the boys").

Pizza Problem Two

The task of sharing one pizza among three boys and three pizzas among nine girls posed no difficulty for Davey who achieved solution using the following strategies.

I. Understanding the Problem. Again, having read the problem, the problem solver ignored the questions and looking at the illustration, he predicted a possible solution to Part A (the boys) (e.g., "Each boy gets about a quarter, I'd say"). The child had planned the solution to Part A (the boys) while making sense of the problem.

II. Solving the Problem. Davey proceeded to then solve the problem by accurately and quickly drawing to partition the circular area into thirds. Pointing to his solution, the researcher probed, "Do you know a name for that?" which elicited an interesting response regarding the child's mathematical knowledge and ability to reason and logically derive a label.

Researcher: Do you know a name for that?

Davey: Three pieces.
Um 1/4 and 1/2 or 1/4. Then I would have --?

Researcher: When you tell me about 1/2, do you know how to write 1/2?

Davey: Yup. A crooked line down. Then you put a 1 on the top and 2 on the bottom.

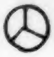
Researcher: Can you show me?

Davey: Like that [1/2]. One half.

Researcher: That's right. Now you've got 1 out of 3 but what are you going to call it? One? Three? What?

Davey: A third. Each boy gets one-third.

Although Davey was not at first able to give the name or illustrate how it might be written, he was able to write one-half, predict one-third would be written in like manner and then correctly label the partition one-third.

To solve Part B (the girls), the child predicted success (e.g., "Oh, it's going to be easy"), monitored the activity, justified the behavior by stating the related detail of the problem and then explained and illustrated the solution (e.g., "Easy, cause see, cause there's three girls right there, three girls right there and three girls there and there's three pizzas so you can divide three like that -- one, two, three pieces []").

III. Reviewing the Problem and the Solution. The solution to each question was then reported to the researcher.

Summary: Language and Strategies Used When Solving Mathematical Problem Two.

1. The problem solver realized two questions were inherent in the problem.

2. The solution to Part A (the boys) was recognized and achieved immediately though the label was not available but was later derived. The solution was to partition in thirds using radial strokes.

3. The solution to Part B (the girls) was achieved by partitioning each of the three pizzas in thirds using radial strokes.

Pizza Problem Three

To solve the problem of sharing two pizzas among three boys and to share eight pizzas among twelve girls, Davey demonstrated the following specific strategies.

I. Understanding the Problem. To understand the problem, the child referred only to that part of the picture concerning Part A (the boys) and the detail was monitored.

II. Planning the Solution. The plan was then stated ("Divide that one into three, that one into three, then you'd give each boy two pieces"). To plan the solution for Part B (the girls), the child monitored the detail, analyzed the solution to Part A (e.g., "I put those upside down Y's") and predicted the procedure could be used to solve Part B (e.g., "Thirds. I'll check it out").

III. Solving the Problem. Monitoring the solution to Part B (the girls) revealed Davey began by using one to one correspondence and then recognized the solution would require assigning "two pieces" to each girl (e.g., "Two pieces each girl, those two, those two, those two"). The child concluded, "Each girl gets two pieces." To the probe, "Is there a name for that?", the child responded by reporting (e.g., "one-third. No. Two one-thirds are a sixth -- six"). As evidenced in the following dialogue, the use of an analogy made to apples by the researcher enabled the child to perceive the label to be two-thirds:

Researcher:	If I put one apple here and one apple here, what have I got?	Davey:	Two apples.
Researcher:	So, if I put one-third here and one third here, what have I got?	Davey:	Six thirds, no, two thirds.

The ability to write two-thirds was subsequently demonstrated by Davey using an analogy to writing one-third (e.g., "So I put two on top instead of one, then you put a three on the bottom like this"). Approval was then sought by showing his work [1/3] and asking a question (e.g., "Like that, right?). Davey then encircled two pizzas and three girls to achieve groups prior to partitioning the circular areas.

IV. Reviewing the Solution. The solution was reviewed by the child by reading and answering each question aloud in sequential order. The task was concluded to be easy and explained (e.g., "Easy, you just put two pizzas to three kids").

In response to the probe, "What did you do?" the child reported and explained that first Part A (the boys) was solved, the solution reflected upon, an analogy made to 'Y,' and the same procedure used to solve Part B (the girls).

> I gave them each 2/3's, and so then I went onto the girls and I started from the back end. Well first of all I started from the front and made all those upside down "Y's" and then I started from the back end and gave two to one girl, and two to the next, two to the next, next, next, next, next.

The predicted solution to Part B (the girls) evidently was based upon the solution to Part A (the boys) rather than details inherent in the problem (e.g., "Well if it worked for the boys, it should have worked for the girls").

Summary: Language and Strategies Used to Solve Mathematical Problem Three.

1. The child proceeded to first solve Part A (the boys) and to derive the label for the solution which was achieved by partitioning the circular areas in thirds using radial strokes. It was recognized each boy would receive two thirds.

2. Projecting the same strategy which had successfully solved Part A (the boys), the child determined the answer to Part B (the girls) but began by first grouping, that is, encircling two pizzas and three girls, then proceeded to partition the circular areas.

Pizza Problem Four

To solve the problem of sharing four pizzas among seven boys and two pizzas among four girls, the following strategies were employed by the problem solver.

I. Understanding the Problem. Ignoring the question, the problem solver sought understanding by means of the pictures and assessed the problem to be easy (e.g., "This is easy").

II. Planning the Solution. To plan the solution, the problem solver first directed attention to Part A (the boys) as evidenced by the comment, "one more boy." A decision to first complete Part B (the girls) was reported (e.g., "I'll start from the girls"). Seemingly, the problem solver planned to halve the pizzas but realized one more boy was required to successfully use the strategy of partitioning by halves.

III. Solving the Problem. As indicated by the language used to monitor the activity Part B (the girls) posed no difficulty (e.g., "Each girl gets half a pizza so two girls get one, two girls get one"). Upon return to the first part of the task, the problem was recognized to be harder than first thought (e.g., "This is harder"). The detail of the problem was again monitored, reference made to the solution for Part B (the girls) (e.g., "If I cut the girls' in half -- except the boys' doesn't work . . .") and the reasoning justified (e.g., "cause if you're cutting all in half -- Let's see, one, two, three -- get a whole pizza"). And again use of the halving mechanism was monitored and an alternative solution recognized (e.g., "two for that one, one big one for that guy. Now you slice off of each"). This alternate solution appears to have given rise to a new proposal based upon real life experience (e.g., "A half --, no -- then you could just put that pizza in a safe place"). The child then reflected upon a differing and final solution also based upon real life experience:

> I've got an idea, then we'll cut this up into seven pieces, this half -- Let's see -- 1, 2, 3, 4, 5 -- this is hard. Four lines -- 1, 2, 3, 4, 5, 6 those are big, 6, 7 there I cut that one into a whole bunch pieces thinking so each boy would get one piece of it. Like that.

Aware that the size of each piece in the shared half was small, the problem solver sought a label by use of logical reason (e.g., "Yup. You give each boy 1/2, 1/2 -- 1/4, a little bit smaller than that. I'd give each boy a little small piece of the piece that was shared"). Further reflection was apparent and a new label expressed (e.g., "Mmmm, a 1/4 of a 1/4"), which was then refined and restated (e.g., "1/2 of a 1/4 of a 1/4").

IV. Reviewing the Solution. When reviewing the solution, the strategies used were reported and explained:

> I figured -- well I did the girls first, cause it was easy, each girl gets a half. Two girls, one pizza split in half there two pieces of pizza, two girls then it goes the same for the other side. Each boy gets half a pizza for this one half pizza, half pizza and he gets that half and he gets one of those and gets one of those and he gets one of those and he gets one of those, he gets one of those, he gets one of those, and he gets one of those.

The label 1/2 of a 1/4 of a 1/4 was simplified (i.e., "one of those"). The process the child explained proved to be one to one correspondence followed by partitioning the remaining half in sevenths.

Summary: Language and Strategies Used When Solving Mathematical Problem Four.

1. The problem was recognized to require two solutions.

2. The second solution, Part B (the girls), was sought first. The strategy used was to group two girls to one pizza then partition in half using a horizontal cut.

3. The first solution, Part A (the boys), was determined by first grouping two boys to one pizza, secondly, partitioning in halves using a horizontal cut and thirdly partitioning the "left-over" half in seven equal parts as shown in Figure 4.8.

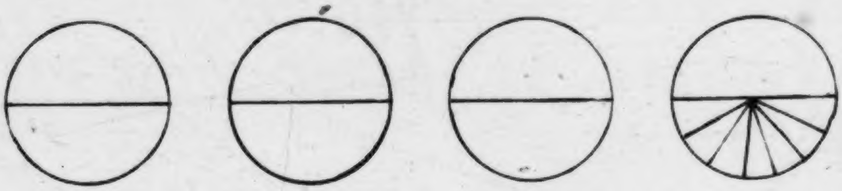


Figure 4.8

Partitioning to Achieve Four-Sevenths

4. The child searched for a label and concluded that each boy received "one half of a quarter of a quarter," that is, one half plus one-sixteenth, and this amount was recognized to be more than one-half, the solution to Part B.

Pizza Problem Five

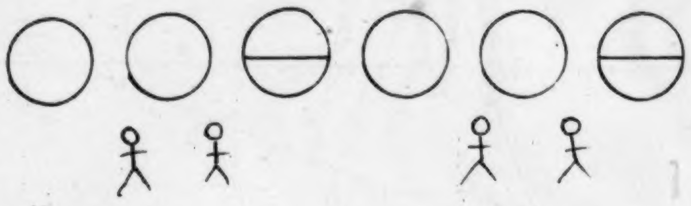
The fifth problem required that the problem solver share six pizzas among four boys and nine pizzas among six girls. To achieve the solution, Davey demonstrated the following specific strategies.

I. Understanding the Problem. To understand the problem, reference was made to only that part of the picture which would be required to answer the first question, "How much pizza does each boy get?" Seemingly the questions were assumed.

II. Planning the Solution. To plan the solution to Part A (the boys), detail was monitored, the problem assessed and the meaning, including the child's feelings, reported (e.g., "Number five -- one, two, three, let me see, four boys. This is simple for the boys"). Lastly, a recognized solution was reported (e.g., "Each boy gets a whole pizza and a half").

To plan the solution to Part B (the girls) attention was directed to the picture by raising a question (e.g., "Okay. Now what about the girls?"). Subsequent steps to plan the solution focused attention to detail (e.g., "How many pizzas? Nine"), redirected attention (e.g., "Let me see how many girls there is"), and monitored the detail (e.g., "So, 1, 2, 4, 6 -- 6 girls. 2, 4, 6 -- 3 pizzas. Oh, oh! There's 2, 4, 6, 8, there's 9 pizza and 6 girls"). The feelings of the problem solver were reflected in the reported conclusion (e.g., "Now it would be easier if there were ten pizzas").

III. Solving the Problem. To answer the first question, the possible solution was monitored (e.g., "see -- that one goes to that one, that one goes to that one, cut in half. That one goes to that one, that one goes to that guy -- goes to that guy"). This monitoring reveals the solution was achieved by use of one to one correspondence followed by partitioning using the halving mechanism and then the process repeated. Also, as evidenced by the activity, grouping had occurred, i.e.,



To answer the second question, the problem solver stated his intent (e.g., "I'll give them each a whole pizza and a half"), monitored the problem solving process (e.g., "That's for her, her, her her, her, her, then -- quarters again. No that one in half; that one to her, that one to her, that --"), and in conclusion reflected upon the meaning of the experience (e.g., "Yup, just give each girl a whole and a half. That was easy").

IV. Reviewing the Problem and the Solution. The review of the solution by the child to himself occurred while solving the problem as each answer to questions 1 and 2 was determined and concluded.

Again by reading and answering aloud each question in sequential order, Davey reviewed the solution to himself.

> A whole and a half. How much pizza does each girl get? A whole and a half. How much pizza, no. If each child gets a fair share of the pizza who would get more pizza, each boy or each girl? All the same. Does each boy get as much as each girl. Yes. How did you figure this out? Easy, I just gave one girl a whole and a half and each boy a whole and a half.

Explanation of the procedure used justified judgement and activity. The explanation indicates that by use of logical reason the child had reflected upon the problem and drawn conclusions. However, the explanation differed from the actual procedure employed; for example,

> Researcher: Why did you give each girl a whole and a half, what made you think to try that?
>
> Davey: Well, I found out that in my head. I didn't know how to split it up. But now I found out. There's two girls for that one, then I found out there's two pizzas left so I gave half to her and half to her, half to her, and half to her. Yeah. That's how I did it.

Apparently, the child recognized two possible alternatives to solve the problem as shown in Figure 4.9.

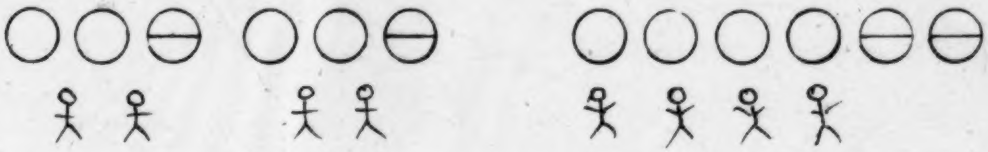


Figure 4.9

Alternative Solutions to Achieve One and One-Half

Summary: Language and Strategies Used to Solve Mathematical Problem Five.

1. The solution to Part A (the boys) was achieved first and labelled as one and one-half.

2. The solution to Part B (the girls) was achieved by first projecting the solution to Part A (the boys) upon the second problem and then partitioning to achieve "one whole and a half" per girl.

Pizza Problem Six

The problem entailed sharing three pizzas among seven boys and one pizza among three girls. The specific strategies used by Davey to achieve the solution are subsequently discussed.

I. Understanding the Problem. Ignoring the questions, the problem solver immediately referred to the picture, first, the boys, then the girls and began to plan the solution..

II. Planning the Solution. Immediate recognition of the solution to Part B (the girls) was apparent (e.g., "One thirds"). To plan the solution to Part A (the boys) the detail was first monitored as if to achieve thirds (e.g., "3, 2; 3, 2; 3, 2 -- goes three pizzas") and two alternate solutions proposed (e.g., 2, 4, 6, 7. Half uh, no, I'll cut it in four halves -- like this") but the size of each of the "four halves" was recognized and reported as "quarters."

III. Solving the Problem. The solution to Part B (the girls) was recognized and immediately determined by partitioning in thirds. The activity to solve Part B (the boys) was accompanied by monitoring as the child partitioned two pizzas in quarters and the remaining pizza in sixths (e.g., "Quarters. Give him one, him one, him one, him one, him one -- give him one of those and split these in half like that. That's for him"). Using language to monitor, the solution was confirmed (e.g., "There. One piece. It's the same size. 3, 4, 5, 6 -- 1, 2, 3, 4, 5, 6, 7").

IV. Reviewing the Problem and the Solution. When reviewing the solution with the researcher, again the solution was revised by monitoring as if to achieve thirds, and by raising a question:

> Because like you give 1/3 to him, one to him, one to him, one to him, one to him, that piece to him -- wouldn't have any. Who would get that whole pizza?
> 1/3. Now I need 1, 2, 3, 4, 5, 6, 7 pieces. 1, 2, 3, 4, 5, 6, 7. There 7. How much does each goy get? Each boy gets one third and a half a 1/4 of a 1/4.

The strategy used was the same as that used to determine the solution to Part A (the boys), Pizza Problem Four. The label "half a quarter of a quarter" to name 1/14 was the same label used to name the solution to Pizza Problem Four, Part A (the boys).

Summary of Language and Strategies Used when Solving Mathematical Problem Six.

1. The child understood two solutions would be required and sought to first determine the solution to Part B (the girls).

2. Part B (the girls): The solution was determined by partitioning in thirds.

3. Part A (the boys); Using alternate possible solutions, i.e., partitioning in thirds, halves, quarters combined with sixths, the problem solver reconsidered grouping three boys to one pizza to achieve partitioning in thirds and then divide the remaining two thirds into seven pieces.

4. The label "one third" was used to name the solution to question 2, Part B (the girls). The label "one third plus a half a quarter of a quarter" was used to name the solution to Part A (the boys).

Part A (the boys): The answer "one third plus a quarter of a quarter" was recognized to be larger than "one third" but the name or label for the fraction was not yet known to the problem solver.

Problem Solving Strategies of the Grade Three Group

The relationship of the members in the group, like that of the grade two children, evidenced friendly equality and mutual support. No one child emerged in a leadership role. Harmony pervaded throughout the problem solving activity but one child at first appeared to more in the role of observer than participant though her infrequent but pertinent contributions reveal active involvement in the group's problem solving process.

Pizza Problem One

To share two pizzas among two boys and five pizzas among five girls was the task required of the group and in so doing, the group demonstrated the following specific strategies.

I. Understanding the Problem. To understand the problem, the group referred to and read aloud the first three questions, then immediately made reference to the picture, Part A (the boys). Further clarification to determine the exact procedure was sought by one group member by directing a question to the researcher (e.g., "Should -- I draw a pizza?").

II. Planning the Solution. By use of prediction, the plan was initiated (e.g., "The boys probably get half a pizza") but the prediction was ignored. A solution was recognized (e.g., "I know a good way to do this") and demonstrated by drawing a ring to encircle one child and one pizza. An alternate method was suggested (e.g., "Just draw a pizza") and agreement concluded by consultation with the researcher that drawing was not necessary as reference could be made to the picture. Attention of the group was directed to Part B (the girls) (e.g., "Now look at the girls"), the solution recognized and stated (e.g., "They each get their own pizza, too") and an explanation based upon observation reported (e.g., "Cause they each get a pizza." "On their heads").

III. Solving the Problem. By encircling each pizza with one child, the solutions to Part A (the boys) and Part B (the girls) were achieved.

IV. Reviewing the Problem and the Solution. To review the solution with the researcher, the children monitored the detail (e.g., Jason: "1, 2, 3, 4, 5, 6, 7 pizzas") and reported the solution (e.g., "One each"). This indicates that although the problem was planned and solved in two sequential steps, i.e., determine how much pizza each boy gets then determine how much pizza each girl gets, the problem was perceived to be all inclusive, i.e., seven pizzas to be shared by seven children. To answer the question, "How did you figure this out?" the procedure was clarified in three steps. First, an imaginative projection, likely based upon past experience, was reported (e.g., Trena: "Maybe they made it"). Secondly, reference was made to the picture (e.g., Marla: "Cause they each get a pizza." "On their heads. They each had their own"). Thirdly, the true procedure was reviewed in a synthesized report based upon logical reasoning and providing direction for the group (e.g., Jason: "We looked at the picture. That's what we should write down. We looked at the boys and the girls and found out they both had a whole pizza"). Note, the detail was again viewed as one problem. Subsequent expression reported the meaning of the solution was likely based upon past experience (e.g., Jason: "What, some hogs! They're pigs!").

Summary of Language and Strategies Used to Solve Mathematical Problem One.

1. Reading the questions aloud, then clarifying the procedure, enabled the group to plan a solution.

2. The problem was solved in two steps: first, the solution to Part A (the boys) was planned; secondly, the solution to Part B (the girls) was planned.

3. The procedure was to encircle one pizza and one child.

4. Though the problem was solved in two steps, the review indicates the problem was viewed in total rather than in two parts.

Pizza Problem Two

Sharing one pizza among three boys and sharing three pizzas among nine girls was the problem posed to the group which demonstrated the following strategies to achieve the solution.

I. Understanding the Problem. Unlike the grade twos, immediate reference was made to the picture in order to understand the problem. The detail of the picture was monitored (e.g., Trena: "one pizza, three boys; three pizzas, nine girls") and the plan initiated to solve Part A (the boys). The detail was again monitored (e.g., Jason: "1, 2, 3, 4, 5, 6, 7, 8, 9") and the plan to solve Part B (the girls) initiated.

II. Planning the Solution. The solution to Part A (the boys) was recognized and the strategy reported (e.g., Jason: "Yeah, I know how. Divide the pizza into three. A third"). The solution to Part B (the girls) was predicted using monitoring and reflecting upon the event to draw a conclusion (e.g., Jason: "That goes to that three. That goes to that three. That goes to that three. So they all get three pieces"). By use of illustration the solution was elaborated, that is, by encircling one pizza and three children to seek agreement (e.g., Jason: "See").

III. Solving the Problem. To solve Part A (the boys), one child, Jason, determined the solution as the others observed. He monitored his achievement (e.g., "Yeah, that's good enough") and made an analogy as he explained the process to the others (e.g., Jason: "It looks like a 'Y'"). The work was concluded by Jason reporting (e.g., "So they all get three pieces [sic]"). The demonstration of the partitioned circle and the analogy made to 'Y' assisted the other children to achieve the solution.

To solve Part B (the girls), the group first encircled each pizza with three girls then partitioned each pizza in thirds always using a 'Y.'

IV. Reviewing the Problem and the Solution. Although initiated by the researcher (e.g., "How much does each boy get"), the review was concluded by the group who read aloud and answered each question in sequential order. The solution (e.g., Jason: "They all get three pieces") was corrected in response to the researcher (e.g., "They all get three pieces?"). The correction (e.g., Jason: "I mean one piece") was justified using reference to the process (e.g., Jason: "You divide all the pizzas into three pieces, so it's a fair share. See!").

The solution was reported again to be "one piece" then renamed, "one third." The question, "If each child gets a fair share of the pizza, who would get more pizza, each boy or each girl?" evoked the following brief discussion.

Language Sample		Language/Thinking Strategies
Jason:	None of them	Reporting a conclusion based upon recognition of the word more.
Jason:	That was sort of easy.	Reflecting upon the meaning of the experience.
Jason:	They each get the same amount.	Monitoring writing the answer.
Marla:	You could just put 'same' instead of all those words.	Directing the activity using a synthesis of the meaning of the solution.
Jason:	Whatever. It think it's fun.	Justifying the activity by reflecting upon one's own feelings.

In response to the question, "How did you figure this out?" an explanation of the procedure was reported (e.g., Jason: "We divided all the pizzas into three").

Summary of Language and Strategies Used to Solve Mathematical Problem Two.

1. By referring to the picture, monitoring the detail aloud, and demonstrating the probable process the group was enabled to proceed to the solution.

2. The problem was solved in two steps: first, the solution to Part A (the boys) was demonstrated and explained; secondly, the solution to Part B (the girls) which had emerged when planning Part A was achieved.

3. The procedure was first to group one pizza and three children, then to partition.

Pizza Problem Three

The problem of sharing two pizzas between three boys and eight pizzas among twelve girls was solved by the grade three group using the following specific strategies.

I. Understanding the Problem. Immediate reference was made to the picture, Part A (the boys), and the solution reported (e.g., Jason: "I know how we do this one. We divide it in half [sic]"). The detail of Part B (the girls) was monitored (e.g., Jason: "1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12. 1, 2, 3, 4, 5, 6, 7, 8") and a solution reported (e.g., Jason: "Divide it in *fourths*. Quarters").

II. Planning the Solution. The possible solutions proposed while endeavoring to understand the problem were sought by the initiator, Jason, but ignored by the other children, one of whom partitioned the circular areas in thirds to solve Part A (the boys) and then reported the solution to the group (e.g., Marla: "Look. Divide it in three. They each get two"). As the discussion developed to determine the solution to Part B (the girls), it appears evident that children are able to use their own knowledge and language to order their experience.

	Language Sample	Language Thinking Strategy
Jason:	Look it. I can make a capital Y. There's three girls that don't get a pizza.	Collaborating and showing work. Makes analogy to explain procedure. Concludes an incorrect conclusion.

Researcher:	Can you look at it another way?	Probe by researcher.
Marla:	Girls are very hungry you know.	Projecting a reason based upon real life experience to disregard partitioning in thirds.
Jason:	Yeah, there's six pieces. She was right about the boys. I wasn't right. You should divide them into fourths. Wait. There's 12 kids. 2, 4, 6, 8, 10, 12, 14, 16. 16?	Refers to solution Part A (the boys). Acknowledges another child's work as correct. Admits own error. Predicts a new solution. Affirms detail and monitors an alternate solution. Questions the possible solution.
Marla:	We don't need 16 pieces. 2, 4, 6, 8, 10, 12, 14. Wait. We don't need that many pieces. So we divide that in half. Try that! 2, 4, 6, 8, 10, 12, 14 and 16. Whew. Divide it in what?	Evaluates the solution. Confirms the evaluation. Predicts the previous solution. Monitors the activity. Raises a question.
Trena:	Three's. 18.	Having attended the dialogue, responds to question and justifies response.

III. Solving the Problem. An imaginative situation based on real life experience accompanied the activity of partitioning in thirds (e.g., Jason: "Isn't this fun chopping up pizzas? They're making me hungry for a pizza. Doesn't it make you hungry?"). Collaboration to approve the solution was sought by showing the work and raising a question (e.g., Marla: "Do you think this is right?"). Before giving approval, the work was monitored (e.g., Jason: "1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12") and a conclusion, based upon apparent recognition of the mathematical fact two twelves are 24, stated (e.g., Jason: "Yup, they each get two pieces").

Attending the monitoring and dialogue though in the role of observer, confirmation and agreement were expressed by the third child ("Yup").

IV. Reviewing the Problem and the Solution. The reported solution to the first question (e.g., Jason: "Two pieces. So it's equal. Half and half"), was clarified by Marla to mean "they all get the same amount" and the use of the word "they" made more precise by Trena (e.g., "each boy and each girl"). To explain the procedure one child, Jason, stated "It's all in the adding" which seemingly prompted a second child, Marla, to read aloud the question,"How did you figure this out?" then report an elaborated explanation of the previous comment (e.g., "I divided the 24 pieces of pizza into the 12 girls. It was easy. In adding. Twenty four pieces into 12 girls. It's all in adding"). Though the term 'divided' is used and two times reference is made to the process of division (e.g., "24 pieces into 12 girls"), Marla concluded "It's all in adding." The researcher's probe, "How?" was acknowledged by Jason (e.g., "By chopping it up in thirds"), and confirmed by agreement (e.g., Marla: "Yeah, thirds"). A discussion which enabled the children to clarify the solution and agree upon a label arose in response to the researcher's probe, "How much did you say they got?".

Language Sample	Language/Thinking Strategy
Trena: One third.	An answer reported.
Marla: Two pieces.	The answer clarified.
Jason: Boy, no, yeah!	Insight.
Marla: Two thirds.	

Jason:	Yeah. A piece is a third.	Restating a fact.
Trena/ Marla:	So another piece is two thirds.	Synthesis.

The brief excerpt of the discussion reveals the students used their language to inform, to clarify, to reason, to justify the reasoning, and to synthesize the content. The contribution of content by each child adds to the understanding of the mathematical concept. Use of the various strategies enabled the children to construct meaning from the content.

Summary: Language and Strategies Used to Solve Mathematical Problem Three.

1. The problem was recognized to have two parts.
2. The solution to Part A (the boys), i.e., two thirds, was first determined. The solution to Part B (the girls) was then determined, i.e., two thirds.
3. The label, two thirds, was derived.

Pizza Problem Four

The strategies demonstrated by the group to solve the problem of sharing four pizzas among seven boys and two pizzas among four girls are presented in the following discussion.

I. Understanding the Problem. Seemingly the questions were assumed as immediate reference to the picture was made by all members of the group and the detail monitored by all (e.g., "1, 2, 3, 4, 5, 6, 7 boys and 4 girls") prior to Jason predicting a possible solution (e.g., "We divide it in thirds again. No halves"). Marla's one utterance (e.g., "Why?"), was used to seek clarification and in response Jason modified and justified the solution using monitoring to explain that the procedure would be appropriate to solve Part B (the girls) (e.g., "For the girls we divide it in half. 1, 2, 3, 4. 1, 2, 3, 4. So --"). The group then endeavored to apply the same understanding to solve Part A (the boys).

II. Planning the Solution. The solution to Part B (the girls) had evolved as the children tried to bring order to the experience. The solution to Part A (the boys) was determined in dialogue as various plans were proposed and rejected as evidenced in the subsequent discussion.

Language Sample	Language/Thinking Strategy
Marla: One boy doesn't -- one boy gets a whole pizza and ends up with --	Reports a conclusion based upon use of the halving mechanism.
Trena: There's one piece left.	Reorganizes the previous information.
Researcher: What else could you do?	Raises a question the intent being to seek an alternate way.
Marla: No! Divide it up in thirds or quarters.	Reports two recognized alternative plans based upon previous experience.
Trena: If you divide it in half, then you get one more piece so that equals eight.	Elaborates the detail inherent in use of the halving mechanism.
Jason: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 -- no. Seven people.	Monitors the detail of the total problem, Part A and Part B. Corrects the monitoring.

Trena:	Seven people but -- yeah, if, if you got 14. If you have 14 each person—each boy would get two pieces. Seven plus seven equals fourteen.	Predicts an alternate goal and justifies the reasoning using a mathematical fact.
Jason:	Divide it in quarters. Oh that will give you loads more than 14.	Reports a procedure to achieve the goal. Recognizes the procedure is not appropriate.
Marla:	Watch. Quarters. I've tried it and it doesn't work.	Demonstrates partitioning in quarters. Concludes the solution should not be used.
Jason:	You sure? -- divided by seven. No.	Questions and confirms conclusion.
Trena:	And gives us one half left over.	Synthesizes and justifies the conclusion.
Jason:	Throw it in the garbage.	Suggests alternate solution based upon real life experience.
Researcher:	Does your Mum do that?	
Trena:	Cut it into seven little pieces.	Recognizes and reports a more appropriate alternate solution based upon real life experience.
Jason:	It wouldn't be fair.	Evaluates alternate solution and reports conclusion.
Marla:	Okay. 3, 4, 5, 6, 7.	Monitors the activity.
Jason:	Hey, that would work.	Recognizes solution.
Trena:	Yeah.	Agrees the solution "works."

Although the language of learning seems disorganized on first impression, examination of the strategies show that the language was used to report, predict, re-organize, monitor, accompany demonstration; that is, explain, question procedure, synthesize and concur. These strategies enabled the children to construct order to the content.

Each child's content adds to the understanding of the proposed solution. The success of the move to order the bits of information into a meaningful content is evident when Marla demonstrates the suggested solution and the group agrees that the solution is feasible. As the solution was planned, the thoughts of the group had been given a more precise shape.

III. Solving the Problem. To solve the problem Part B (the girls), the group first partitioned the circular shapes in halves using horizontal cuts. This same procedure, use of the halving mechanism, initiated the solution to Part A (the boys). The second step was to then partition the last half in "sevenths" using radial spokes. It is interesting to note partitioning in sevenths did not begin with a vertical radius, i.e., quarters, but was determined by use of six radii. The problem solving procedure was monitored as the remaining half was partitioned and each slice numbered.

IV. Reviewing the Problem and the Solution. The solution to "How much pizza does each boy get?" was simultaneously reported by all (e.g., "Two pieces"), then revised (e.g., Jason: "No. One. No, one and one seventh -- and -- one half and a seventh") and a reflection upon the meaning of the solution reported (e.g., Jason: "And that wouldn't be fair because the girl isn't getting a seventh"). The solution to Part A (the boys) was again revised by Marla (e.g., "A half and a teensy bit. A half and a teensy bit") and the "teensy bit" labelled "a seventh" by Jason. Again, the meaning of the solution was reflected upon by Jason (e.g., "That wouldn't be fair to the girls"). Nevertheless, the children acknowledged, "the boys get more because he [sic] gets a half and a seventh and they [the girls] only get half." To explain "How did you figure this out?" only the procedure to solve Part A (the boys) was reported (e.g., "We divided one pizza in half and a seventh") and subsequent language monitored the written response to answer the question, "How did you figure this out?"

Summary: Language and Strategies Used to Solve Mathematical Problem Four.

1. The group referred directly to the picture and after some confusion due to looking at the entire problem, the solution to Part B (the girls) was recognized and reported to the group. Then the solution to Part B (the girls) was achieved by partitioning in halves using horizontal cuts.

2. A series of partitioning strategies, predicted as possible solutions based upon stated mathematical facts (e.g., 7 + 7 = 14) and discussion of the facts initiated the use of the halving mechanism and progressing through partitioning by thirds, quarters, then combining thirds to achieve two-thirds. As each strategy was discussed, attempted and rejected, the group again resorted to use of the halving mechanism. Seemingly, the group had endeavored to initially use partitioning by halves to solve Part A (the boys) because this solution had "worked" to solve Part B (the girls) and the overriding thought was to be fair to all rather than to achieve fairness within each of the two sub-problems.

3. The final solutions involved partitioning in halves to solve Part B (the girls) and then by first partitioning in halves followed by partitioning one half in sevenths to achieve the solution to Part A (the boys). The seven shares were then numbered as indicated in the following figure.

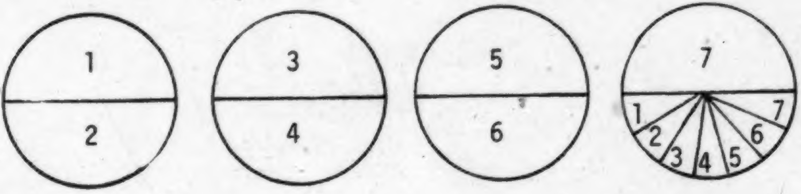


Figure 4.10

Partitioning Four Sevenths

SOLVING THE PIZZA PROBLEMS: GRADE FOUR

The problem solving strategies of an individual grade four child and a small group of grade four children will be specifically described in the following discussion.

Problem Solving Strategies of the Individual Grade Four Child

Unlike either of the individual grade two and three children, the individual grade four child initially and throughout the problem solving process manifested much frustration, sighing and tears. Thus the researcher was prompted to clarify and direct the child's thinking by means of questioning in order that the child be enabled to achieve some success. Therefore the child in this part of the study is very dependent.

Pizza Problem One

When presented with the task of sharing two pizzas between two boys and five pizzas among five girls, the following strategies were demonstrated by the child.

I. Understanding the Problem. To understand the problem, one child silently read the questions, referred to the picture and reported (e.g., "I don't know how to do this"). The apparent lack of understanding and endeavor to further pursue the problem prompted the researcher to assist the child to make some meaning of the problem and ultimately to plan the solution.

II. Planning the Solution. The first question was read aloud to the child by the researcher and the activity directed by questioning (e.g., "Can you tell by looking?"). The response, tears, evoked further questions (e.g., "How many boys are there?" "How many pizzas do they share?" So how much is that each?"). The answers reported in sequential order were "two," "two," and "about eight pieces each." This seems to indicate that possible confusion arose from past experience, i.e., whole pizzas are cut in about eight pieces. Reference was made to the picture, directing the activity (e.g., "But if you just look at the picture"), the detail in the problem stated (e.g., "If there's two boys and two pizzas"), and the question rephrased ("How much pizza does each boy get?") and the elicited response was "just one pizza." To assure the child of her capability and to satisfy the curiosity of the researcher, that is, will the answer be about four pieces or one-half, an aside, posing a new problem, was introduced (e.g., "Supposing there was only one pizza there, how much pizza would each boy get?") and the reported response was "one-half." Returning to the original problem (e.g., "Okay. You're good at this. Now, where it says, 'How much pizza does each boy get?' did you decide?"). The answer reported was "eight pieces -- or one whole pizza" and the process desired to answer the question clarified (e.g., "printing or writing"). The solution to question two, "How much pizza does each girl get?" was derived quickly and confidently using reference to the picture and with no support from the researcher and the reasoning explained (e.g., "Because there's five pizzas and five girls").

IV. Reviewing the Problem and the Solution. In response to the question, "Who would get more, each boy or each girl?", the child maintained that each girl got more, despite the review and the reported solutions remaining unchanged, that is, "each boy gets one whole pizza" and "each girl gets one whole pizza." She explained this conclusion was arrived at "by looking at like how many girls and how you can divide them."

Pizza Problem Two

To achieve the solution to share one pizza among three boys and three pizzas among nine girls, the problem solver used the following specific strategies.

I. Understanding the Problem. No reference was made to the questions as the child immediately referred to the picture, Part A (the boys), and reported a possible solution (e.g., "Just divide it into three and each boy will get one piece"). Reference was then made to the picture, Part B (the girls), and a possible solution reported (e.g., "You need to put 24 pieces on these three pizzas").

II. Planning the Solution. Part A (the boys): The plan evolved as the child sought to understand the problem but her unsuccessful attempts to partition in thirds and the report, "I can't" prompted the researcher to suggest practice on a larger circular area might benefit (e.g., "Would you like to practise on a larger pizza?"). The suggestion was acknowledged (e.g., "Uh huh"), and the child was able to successfully partition in thirds a circle two inches in diameter. The procedure was later transferred to the smaller circular areas when solving the problem.

Part B (the girls): The predicted solution to put "24 pieces on these three pizzas" was revised, likely due to the small size of the circular areas (e.g., "Except I really can't do that. That's not right. Oh, I know how to do it. You just put eight pieces on each pizza"). By raising a question (e.g., "Why would you get eight?"), the child was encouraged to explain her thinking (e.g., "Cause 3 x 3 is -- cause there's eight"), correct the detail (e.g., "No, there's nine girls"), state a conclusion (e.g., "That won't work") and predict a new possible solution (e.g., "Maybe I'll have to divide it into 27").

III. Solving the Problem. Part A (the boys): Having partitioned a larger area in thirds, the procedure was successfully transferred to the smaller area but the process was predicted to be harder (e.g., "I'll try. This is going to be harder"). The use of the phrase, "I'll try" indicates uncertainty regarding one's own capability.

Part B (the girls): The child sought to partition the first pizza in ninths initiating the procedure by using the halving mechanism. This procedure was recognized to be inappropriate (e.g., "No, I've got 10 --") giving rise to a new procedure (e.g., "Make the pieces bigger"), while the process was monitored aloud (e.g., "I think I just did it. I've just got to put this one in the middle. There"). The difficulty of using such a procedure, i.e., partitioning in 10, erasing and juggling nine radii, provoked the researcher to further question the child, hoping to clarify the thinking (e.g., "Okay, if you're making nine, what might be a good place to start? What might you do first?"). The response (e.g., "Hmm. Oh yeah! Three's") indicates reflection upon the meaning of the experience to glean new insight but not to perceive the solution as a total of nine pieces. By first partitioning in thirds, then by adding further radii to achieve ninths, i.e., 27 pieces, the solution was determined.

IV. Reviewing the Problem and the Solution. The process used to review the solution was one of question and report. The solution to the question, "How much pizza does each boy get?" was reported as "one piece." In response to the probe to determine possible knowledge of a label (e.g., "There's a name for that. Do you know the name of that piece?"), the child stated, "one quarter." Subsequent review of the solution and recognition that each boy gets "one piece out of three" evoked the probe, "Do you know a name for one out of three?" and the response (e.g., "I don't think -- I learned it before but I think I forgot"). Apparently the child realized "one out of three" was not "a quarter" but lacked a label to identify her answer. Nevertheless the solution was correct.

The solution to the question, "How much pizza does each girl get?" was reported as "one piece each of nine pieces" and as "three pieces out of 27"). By raising a question, the child sought further clarification (e.g., "How do I do this bottom one, the one about how did you figure this out"). The response of the researcher (e.g., "How did you figure it out?") provoked explanation based upon logical reasoning to justify the procedure:

> I just counted the boys and the pizza and I figure that would be three pieces for the boys but I had to use multiplication to do the girls. Well I had to go like 3 x 9 girls and that's 27 so I knew I had to have 27 pieces.

Reflecting upon the question posed by the researcher (e.g., "Could you have done it another way, without multiplying?"), an alternative solution was recognized and reported (e.g., "Three pieces on each pizza"). The solution "Three pieces each" was reported and in response to the question, "Supposing you had divided the other way that you told me about, then how much pizza would each girl get?", the solution was restated and revised (e.g., "One piece each," "one piece out of nine," "nine pieces in all the pizzas together," "there would be three pieces in each pizza," "one out of nine? Well, one out of three"). This procedure enabled the child to perceive and conclude that each boy and each girl would get "the same," and report an explanation which justified the conclusion (e.g., "because I divided them the same way").

Summary: Language and Strategies Used to Solve Mathematical Problem Two.

1. The child reported the solution to Part A (the boys) would be to divide the pizza into three but no label was available and considerable difficulty was experienced when trying to accomplish the activity, that is, partitioning in thirds.

2. The final solution to Part B (the girls) was based upon the child's knowledge of the mathematical facts, 3 x 3 = 9 and 3 x 9 = 27. The solution was reported as one piece out of nine (per pizza) and as three pieces out of twenty-seven. The child was able to report a second possible solution, that is, to partition each circular area in thirds but again no label was available.

3. The child was able to deduce from the second possible solution to Part B (the girls) that, in fact, each boy and each girl had received the same amount though the conclusion was based upon the reason, "I divided them the same way" which was in fact not so.

Pizza Problem Three

The child demonstrated the following specific strategies while endeavoring to share two pizzas among three boys and eight pizzas among twelve girls.

I. Understanding the Problem. Ignoring the questions, understanding was sought through reference to the picture, Part A (the boys). As no understanding was apparent, the researcher sought to clarify the child's thinking by means of questioning (e.g., "What would you do if there was only one pizza there?"), which evoked the response "I would divide it into three pieces." The probe, "So then you get another pizza?" seemingly ordered the child's thinking enabling recognition of the solution (e.g., "Six pieces. You just put six pieces, well you just put two pieces, like three pieces in each of these").

Reference was again made to the picture, Part B (the girls), and the detail monitored (e.g., "There is 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 girls and there is 1, 2, 3, 4, 5, 6, 7, 8 pizzas"). Raising a question (e.g., "Now what?") seemed to spur the child to devise a plan.

II. Planning the Solution. Part A (the boys): Understanding that it was possible to achieve "six pieces" enabled the child to plan the solution (e.g., "And then you give each boy two").

Part B (the girls): Using language to monitor the detail in conjunction with use of timetable facts to determine a possible solution (e.g., "2 -- 16. That's not enough. [Inaudible self-talk.]

I figured it out that they get two pieces each"), seemingly provided a problem solving model as evidenced in the explanation (e.g., "Like I counted. I went 2 x 8 is 16 but that wasn't enough, and then I counted 3 x 8 is 24 so then they get two pieces each").

III. Solving the Problem. The solutions were achieved by partitioning all pizzas in thirds.

IV. Reviewing the Problem and the Solution. In the review of the problem, it was reported that "each boy gets two pieces each" and each girl would get the same amount, that is, two pieces.

Pizza Problem Four

The task of sharing four pizzas among seven boys and two pizzas among four girls seemed to require the child to undertake the following specific strategies.

I. Understanding the Problem. Immediate reference was made to the picture, an observation reported (e.g., "Hmm. There's more boys than there are girls"), the detail monitored (e.g., "There's 1, 2, 3, 4, 5, 6, 7, 8 [sic] boys and four pizzas"), and a mathematical fact stated (e.g., "4 x 7 is 28").

Continuing to look at the picture, the child recognized and reported the plan which could be used to solve Part B (the girls), i.e., halving.

II. Planning the Solution. First, the plan to solve the question "How much pizza does each girl get?" was reported (e.g., "Well, if I cut it in half then these two share that pizza and these two share that pizza"). By pondering aloud the child proposed a plan by which the solution could be derived to solve Part A (the boys) (e.g., "Twenty-eight. Cut them in four. No. Cut them into seven").

III. Solving the Problem. To solve the problem partitioning in sevenths by use of radial cuts was first practised on a larger circular area. Then the behavior was transferred to the original page presenting the problem. The activity was directed by monitoring and reporting (e.g., "There. I'm finished them all"). No speech accompanied the solution to Part B (the girls) which was achieved by using the halving mechanism, a horizontal cut.

IV. Reviewing the Problem and the Solution. As the solution was reviewed with the researcher, it became apparent that the problem solver was unable to answer the first question even though having achieved the correct solution by partitioning. Neither awareness of the number of pieces per child nor a label were apparent. By use of question and answer as evidenced in the following dialogue, the activity was reviewed hoping the child's information would move toward order.

Researcher Questions	Suzanne's Responses
If you only had one pizza, how much would each boy get?	One piece each.
Out of how many?	Four
Did you cut it in four?	No.
So he gets one out of how many?	Seven.
But you haven't got one pizza, you've got four.	So it would be three pieces each.

Don't they get any from the fourth pizza?	Yeah -- four pieces each.

The process of question and answer enabled the child to arrive at a concept, "four pieces," which would enable comparison to be made with the solution to Part B (the girls) (i.e., "One piece each," "one half"). To determine "Who got more, each boy or each girl?", the problem solver looked at the pictures and reported, "each girl." Again the researcher probed (e.g., "So half is bigger? Is that what you're telling me? Okay. So does each boy get as much as each girl?"). It provoked a comparison and the report (e.g., "Well four pieces would probably add up to one half, so they would probably get the same"). A dialogue pertaining to the meaning of one-half (e.g., "What's 1/2 of 2?", "1"; "What's 1/2 of 4?", "2"; "What's 1/2 of 6?", "3"; "What's 1/2 of 8?", "4") shed no light upon the conclusion to the next question which was read and answered by the child (e.g., "If each child gets a fair share of the pizza, who would get more, each boy or each girl? The girls, because they get a half and these guys only get four pieces -- out of four"). Apparently, regardless of having executed the solution, the child did not, in fact, derive meaning from the process, the picture or dialogue, nor was there an available label to more precisely describe the solution other than "pieces."

Pizza problems five through eight were not presented because, for this child, the tasks seemed to be much too arduous.

Problem Solving Strategies of the Grade Four Group

Like the grade two and grade three groups, the relationship of the members of the group was friendly, harmonious and mutually supportive.

Pizza Problem One

Given the task of sharing two pizzas between two boys and five pizzas among five girls, the group evidenced the following specific strategies.

I. Understanding the Problem. To understand the problem, immediate reference was made to the picture.

II. Planning the Solution. No plan was stated.

III. Solving the Problem. To solve the problem, the group first focused attention only to that part of the picture concerning Part A (the boys) and simultaneously reported the solution (e.g., "One pizza each"). This process was repeated to solve Part B (the girls). No other activity was evidenced.

IV. Reviewing the Problem and the Solution. All three children simultaneously read and answered aloud each of the first three questions. The fourth question, though read by all, was answered by one child:

> William: Two pizzas and two boys and then each boy gets a whole pizza and then there's five more pizzas and there's five more girls so they don't have to divide it cause you can see that they each get a whole pizza. Instead of having to divide it into pieces - they each get a whole pizza.

The response was confirmed by the others (e.g., "Yeah") and then the answer elaborated (e.g., Brenda: "Like if there was one pizza you would have to divide it into five pieces - maybe six and one left over but here they each get one whole pizza").

Summary of Language and Strategies Used to Solve Mathematical Problem One.

1. Reference to the picture was made to understand, plan and solve the problem. As evidenced in the review of the problem, the picture was understood to have embedded two separate questions.

2. While reviewing the problem, the children reported that detail had been recognized, relationships perceived, the conclusion made that division was not thought to be necessary and the conclusion justified.

Pizza Problem Two

To the given task of sharing one pizza among three boys and three pizzas among nine girls, the grade four group demonstrated the following specific strategies.

I. Understanding the Problem. To understand the problem, reference was immediately made to only that part of the picture concerning Part A (the boys) and then the attention focused upon only that part of the picture concerning Part B (the girls).

II. Planning the Solution. To plan the solution, the group immediately recognized and made the prediction that partitioning in thirds would solve both parts of the problem (e.g., "Divide it into three").

III. Solving the Problem. Each child quickly and successfully partitioned each pizza in thirds by first determining the centre and then using either a 'Y' or inverted 'Y.'

IV. Reviewing the Problem and the Solution. The group read aloud each of the first three questions and briefly reported the answers (e.g., "a third each," "a third each," "neither: the same amount"). The label "equal" was used to elaborate "the same amount." After reading the question, "How did you figure this out?", by means of discussion the group elaborated and concurred upon the procedure used to achieve the solution to Part A (the boys):

> Brenda: We divided it into one-thirds. Like the boys got one-third because they wouldn't all get the same amount if you cut it into four pieces. Then there would be one piece left over. And if a piece is left over, they'd probably be arguing.

In this discussion, the decision was justified by use of a logical reason supported by reference to past experience and by raising a question (e.g., "Who'd get the last piece?"). In further discussion to review the procedure in order to determine the solution to Part B (the girls), reference was made to detail (e.g., "There's three pizzas and 12 girls"; the detail was corrected (e.g., "No, nine girls") and the procedure explained by use of logical reasoning.

> William: And if the boys got one-third of a pizza each and there's three pizzas for the girls then divide the girls into thirds and each third of the girls gets one pizza. Then you divide that pizza into the thirds like the boys.

Summary: Language and Strategies Used to Solve Mathematical Problem Two.

1. Seemingly the problem was assumed to have two parts and immediate reference was made to the picture, Part A (the boys), and then to Part B (the girls).

2. By partitioning in thirds, the solution to Part A (the boys) was achieved. To find the solution to Part B (the girls), the procedure used was to first divide the girls into three groups, each group corresponding to a pizza and then to partition each pizza into thirds using either 'Y' or an inverted 'Y.'

Pizza Problem Three

The task of sharing two pizzas among three boys and eight pizzas among twelve girls was evidenced in the following specific strategies.

I. Understanding the Problem. To understand the problem, the group first made reference to the picture, then to only the first question which was read aloud by Brenda to the group (e.g., "How much pizza does each boy get?").

II. Planning the Solution. To plan the solution to Part A (the boys), a prediction based upon reference to detail, and a recognized relationship was proposed by one child, Mathew, to the group:

> I think I have a solution for this. Sounds a little wacky but -- Since there's only three boys and two pizzas these boys could get two-thirds. Like you can make it into thirds and then this boy gets one-third from this pizza and one-third from that.

The use of the expression "I think" suggests the child was aware other possible solutions might exist. The expression "sounds a little wacky but --" seems to reflect the child's own feelings regarding the meaning of the procedure and, at the same time, help the group attend to the report.

The plan to solve Part B (the girls) was initiated by William asking a question (e.g., "Now, what about the girls?") and as the group talked it through they recognized the solution to be the same as used to solve Part A (the boys).

William: Three girls and two pizzas.
All: Okay there's 12 girls.
Brenda: But you see --
William: Three girls to two pizzas.
Mathew: So the same as the boys.
Brenda: Yup.

III. Solving the Problem. To solve the problem, the children first achieved the solution to Part A (the boys). While monitoring aloud the activity, each child partitioned each pizza in thirds and the stated conclusion "each child gets two pieces" was restated using mathematical terminology (e.g., "Each piece is one-third but they each get two-thirds in all").

To obtain a solution to Part B (the girls), the children had monitored the activity while partitioning each pizza in thirds. (e.g., Brenda: "This piece. This piece. And this piece for the third girl"), reviewed the detail (e.g., William: "Three girls and two pizzas"), recognized the relationship (e.g., Mathew: "Three girls to a pizza"), and concluded the solution (e.g., Mathew: "So the same as the boys." "Each girl gets two-thirds of a pizza") and agreed upon their solution (e.g., Brenda: "Yup").

IV. Reviewing the Problem and the Solution. To review the solution, the group first read aloud, then answered each question. In response to the question, "How did you figure it out?", the use of logical reasoning to explain the procedure to solve Part A (the boys) was evident (e.g., Brenda: "Figured it out to split it up into three thirds and then if one boy got one piece of pizza, then the other boy got one and the other boy got one"). The explanation was then elaborated by William (e.g., "There would be three pieces left over") and agreed upon (e.g., Brenda: "Yeah"). The solution to Part B (the girls) made reference to past learning, that is, knowledge of even and uneven numbers (e.g., William: "There's nine girls. An uneven number. And an even number of pizzas"). In response to the probe, "What made you decide to try thirds on the girls?", again knowledge of even/uneven numbers was reported (e.g., "Well, because there's sets of three on the girls and sets of two on the pizzas"). Apparently recognition of "sets" helped the group to achieve the solution.

Summary: Language and Strategies Used to Solve Mathematical Problem Three.

1. By referring to only that part of the picture concerning the boys and talking through the detail to arrive at the solution, the group was able to plan and achieve the solution for Part A (the boys). The solution entailed partitioning in thirds and the label two-thirds was used to identify the answer.

2. By referring to only that part of the picture concerning the girls, and talking the problem through, the solution to Part B (the girls) was sought and achieved by grouping in "sets" based upon knowledge of even and uneven numbers. The "sets" were recognized to be the same as in Part A (the boys), that is, each set contained three girls and two pizzas. The pizzas were partitioned in thirds using 'Y' or inverted 'Y' (λ) and, to name the size of each share, the label two-thirds was used.

3. The activity gave rise to the children illustrating their solution as evidenced in the following figure.

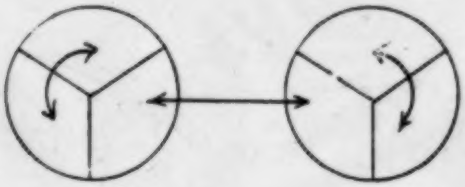


Figure 4.11

Partitioning by the Grade Four Group to Achieve Two-Thirds

Pizza Problem Four

The task of sharing four pizzas among seven boys and two pizzas among four girls elicited the following behaviors:

I. Understanding the Problem. To understand the problem, the children first looked at the picture (e.g., William: "Oh. Look!"), reflected upon the experience (e.g., Mathew: "Geesh"), decided to first solve Part B (the girls) (e.g., William: "Do the girls first") and justified the decision (e.g., Mathew: "Yeah. They're the easy ones"). Likely past, real life experience, that is, sharing and talking about halves, accounts for the group perceiving this part of the problem as easy. To understand Part A (the boys) the children first attempted to partition in halves, that is, to apply the solution of Part B (the girls). The group concluded partitioning in halves would be inappropriate (e.g., William: "Won't work the same"), and then attempted to partition in thirds (e.g., Mathew: "I think this will work"), but recognized this too would not solve the problem (e.g., Mathew: "No, it wouldn't"). The question was raised by Mathew, "Could everybody get the same amount?" and the children again attempted to partition in halves and again found "that won't add up the same," i.e., the same as the girls. An alternate plan was suggested by William (e.g., "We haven't tried fifths yet"). Again, the detail was monitored and Mathew expressed new insight (e.g., "I have an idea. Put it into sevenths"). The suggested procedure was justified (e.g., "Put it into sevenths -- so it's seven pieces"). However, this justification was rebutted by William (e.g., "Uh Uh! You can't divide an uneven number, seven, into an even number, four"). By raising a question Mathew sought to clarify the explanation (e.g., "Why not? You can divide it. You could do that. Then they each get four pieces"). The relationship of the detail was reported by Brenda (e.g., "There's seven boys and there's four pizzas"), provoking further explanation and extracting the central meaning of the problem by use of logical reasoning (e.g., Mathew: "I know. Divide each one into seven and then they each get four pieces"). The interruption (e.g., William: "That won't work. There's an uneven number of boys. There has to be six boys"), followed by the response (e.g., Mathew: "That's with numbers, not real things") and William's subsequent agreement (e.g., "Oh. Oh yeah") seems to indicate an assumed reasonableness of the statement which had distinguished between numbers and "real things." By raising a question (e.g., Brenda: "Yeah, but what about the other pizzas?"), further clarification was sought and the plan elaborated (e.g., Mathew: "Do it the same way"). The group concurred in the decision (e.g., William: "Yeah, they each get four-sevenths of a pizza").

II. Planning the Solution. The plan had evolved as the children sought to understand the problem.

III. Solving the Problem. Immediate recognition to Part B (the girls) was accompanied by the activity of partitioning in halves using a horizontal cut.

To achieve the solution to Part A (the boys), each pizza was partitioned in sevenths using radial cuts. The activity was directed by use of monitoring (e.g., "1, 2, 3, there, there, there, 1, 2, 3, 4, 5, 6, -- there, 7") as each child partitioned the pizza into seven pieces of equal size and the conclusion reported (e.g., William: "Yeah; they each get four-sevenths").

IV. Reviewing the Problem and the Solution. To review the solution, the first three questions were read and answered aloud in sequential order by all (e.g., "Four-sevenths, one half. Each boy gets more"). By use of logical reasoning to explain the procedure, the group answered the question, "How did you figure this out?":

Brenda: We counted out how many boys there was. Then we counted how many pizzas and there's four pizzas and seven boys as so we divided them into seven --

Mathew: And there's four pizzas and we did that on all four pizzas so each boy would get four pieces -- four-sevenths.

William: And for the girls we split it in half.

Brenda: Cause there's two pizzas and four girls.

In response to the researcher's probe, "How did you figure out who had more?", the children reported, "Well four-sevenths is more than a half," then justified the answer by use of logical reasoning (e.g., "Cause four is more than three and three takes up less than -- well the four must go into the half way mark of seven").

Summary: Language and Strategies Used to Solve Mathematical Problem Four.

1. The group immediately recognized and stated the solution, one half, to Part B (the girls).

2. The group arrived at the solution, four-sevenths, to Part A (the boys) by talking through and rejecting varying possible solutions based upon past experience and the mathematical concept of even/uneven numbers. The correct solution finally gleaned was elaborated and the necessary procedure explained prior to its demonstration. Through their dialogue the group had reworked the meaning of the problem, elaborated the meaning of the procedure and established the correct label and its meaning to identify the solution.

3. The group was able to determine four-sevenths was more than

one half by talking through the meaning of one half.

Pizza Problem Five

The problem of sharing six pizzas among four boys and nine pizzas among six girls was solved using the following strategies.

I. Understanding the Problem. Ignoring the questions, the group referred to the part of the picture concerning Part A (the boys) and monitored the detail aloud enabling them to plan the solution. The same procedure was repeated in order to determine the solution to Part B (the girls).

II. Planning the Solution. Having monitored the detail to Part A (the boys), the plan was initiated (e.g., Mathew: "Split it into six. Then they each get four pieces -- they each get six pieces"). The initiated plan was acknowledged and the process represented by William who seemed to suggest the desired procedure would require that the six pizzas be grouped to achieve four equal shares (e.g., "It's almost just the other way around"). By raising a question, including reference to the detail inherent in the problem, Mathew reflected upon William's suggestion (e.g., "There's four boys and there's six pizzas so you split them into -- fourths?") but the possible solution was recognized as tentative (e.g., "So they get -- let's see"), then confirmed by Brenda using monitoring (e.g., "It works").

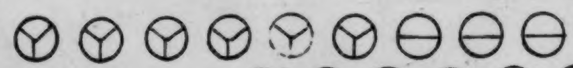
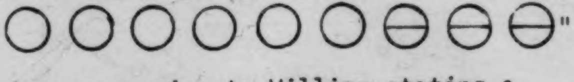
To plan the solution to Part B (the girls), the detail was monitored by each child and reference made by William to a possible pertinent mathematical fact (e.g., "Nine divided by six is three. No!")

which was clarified by Mathew (e.g., "Remainder three. Nine divided by six is one remainder three"). The recognition of the mathematical fact gave rise to two alternate solutions.

III. Solving the Problem. Part A (the boys): Posed solutions based only upon observation (e.g., William: "They each get a half of a pizza," and Mathew: "They each get one and a fourth") were clarified by Brenda who, having monitored her activity, that is, partitioning in fourths, reported and explained the solution (e.g., "Split it in fourths then each one gets six"). William acknowledged the report and synthesized the solution (e.g., "Each boy gets a whole pizza and a half"). Brenda then questioned if another method existed (e.g., "Is there another way?"). The solution was rephrased to be "six fourths" or "six quarters" by William and Mathew.

Part B (the girls): Brenda, having monitored the detail (e.g., "1, 2, 3, 4, 5, 6, 7, 8, 9 girls and 1, 2, 3, 4, 5, 6 -- 6 pizzas"), reported a possible solution (e.g., "I did it, I think. Each girl gets four pieces of pizza"), and then explained the process (e.g., "Because you give the girls each a pizza first and then the other three pizzas divide in halves"). The solution provoked a discussion using logical reasoning to analyze (e.g., William: "Two-thirds [sic]. each person gets three thirds of a pizza. One whole"), synthesize the solution (e.g., Mathew: "They get a pizza and a half"), and Mathew then justified and elaborated the synthesis:

> Instead of splitting these up in thirds and then the remaining three you split in half -- give them a whole pizza. Cause there's only three left and there's six girls so you split them in half so there's six pieces. So each gets an extra piece.

A comparison of the procedures by way of illustration ensued (e.g., Brenda: "Look "; Mathew: "You have the same thing. See ") The discussion and use of comparison gave rise to William stating a mathematical principle (e.g., "Yeah, they can have three thirds and a half or one pizza and a half. It's the same thing").

IV. Reviewing the Problem and the Solution. To review the solution the group read and answered aloud each question in sequential order. The mathematical concept six-fourths was named three ways: "six quarters," "six fourths" and "one and a half" and the mathematical principle stated by the group (e.g., "Six quarters equal six fourths equal one and a half. It's all the same"). To answer the question "How did you figure it out?", the group seemingly made reference to the picture while reporting their own feelings (e.g., "Did the boys first 'cause the girls were harder").

Summary: Language and Strategies Used to Solve Mathematical Problem Five.

1. It was recognized that the problem would require two solutions.

2. The solution to Part A (the boys) was achieved by first partitioning in quarters and the solution six-quarters equated to "a whole and a half." The solution to Part B (the girls) was achieved by two differing procedures as evidenced in Figure 4.12.

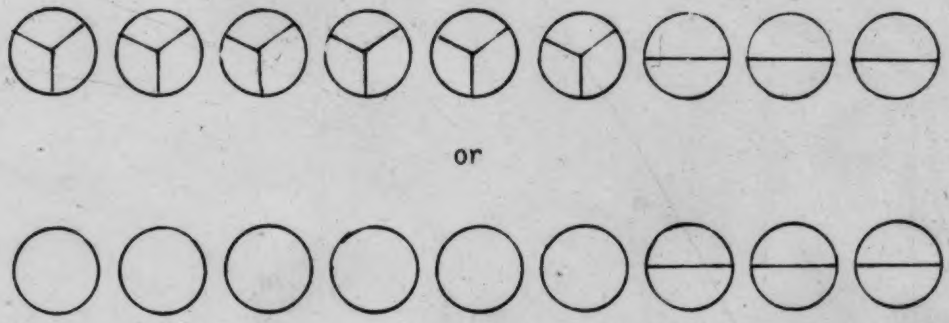


Figure 4.12

Partitioning to Achieve One and One-half

3. It was recognized and stated that the mathematical concept three thirds and half is one whole and a half.

With the intent to eliminate confusion arising from the juxtaposition of understanding the problem and planning a solution, these two steps, Understanding the Problem and Planning the Solution are presented under one heading throughout Pizza Problems Six, Seven and Eight.

Pizza Problem Six

The problem of sharing three pizzas among seven boys and one pizza among three girls was solved using the following strategies:

I. Understanding the Problem and Planning the Solution. To understand the problem, reference was made to the picture Part A (the boys), a reflection of the meaning of the experience based upon one's own feelings reported (e.g., Mathew: "This is easy") and the plan stated by William (e.g., "The girls -- thirds").

Subsequently, as the detail was monitored to Part B (the girls), the group determined the plan in three steps: (i.e., "Cut each pizza in seven"; "Give each boy three pieces"; "Three sevenths. Each boy gets three sevenths").

II. Solving the Problem. The solution to Part B (the girls), partitioning in thirds, was drawn quickly and accurately by all. Neither monitoring nor intercommunication was evidenced.

The solution to Part A (the boys), partitioning in sevenths using radial cuts was accompanied by inaudible self talk but no difficulty was apparent.

III. Reviewing the Problem and the Solution. To review the solution, each question was read and answered aloud by the group. To determine which was more they then referred to their work (e.g., "Look at the picture"), and reported a conclusion with reservation (e.g., "Three-sevenths is more. Well, a little. The boys get more"). The response to the question "How did you figure this out?", was a report of the experiental meaning (e.g., "Easy") and a summation of the procedures (e.g., "Thirds for the girls. Then cut in seven for the boys").

Summary: Language and Strategies Used to Solve Mathematical Problem Six.

1. The problem was known to require two solutions.

2. The first solution achieved was determined by partitioning in thirds and the second by partitioning in sevenths. A label was available to identify each solution.

3. A comparison based upon observation and analysis of the solution enabled the group to determine three-sevenths was more than one-third. No diagrammatical strategy other than partitioning was used.

Pizza Problem Seven

The problem of sharing five pizza among three boys and eight pizzas among five girls was solved using the following specific strategies.

I. Understanding the Problem and Planning the Solution. To understand the problem, the children first referred to the total picture, then agreed to first do Part A (the boys) (e.g., "Do the boys first"). After monitoring the detail, the possible solution was reported by Mathew (e.g., "Give each boy one pizza. Cut the other two [pizzas] in thirds. Then give each boy two pieces. Each boy gets one and two-thirds").

To solve Part B (the girls), first the detail was monitored, then the solution proposed by William (e.g., "Give each girl one pizza. Cut the remainder in five, fifths. Give each girl three pieces"). The meaning of the solution was clarified by Brenda and Mathew (e.g., "Each girl gets one pizza and three pieces"; "One pizza and three-fifths").

II. Solving the Problem. The solution to Part A (the boys) was achieved by partitioning two of the circles in thirds using radial cuts. The solution to Part B (the girls) was achieved by partitioning three of the circles in fifths using radial cuts. No intercommunication occurred but inaudible self-talk was evident.

III. Reviewing the Solution. To review the solution, each question was read and answered aloud in sequential order. The solution to question 1, "How much pizza does each boy get?" was reported to be "one and two-thirds," then elaborated, "Yeah, five-thirds too!" The solution to the question "How much pizza does each girl get?" was reported to be "one and three-fifths." To determine which was more, one and two-thirds or one and three-fifths, a dialogue based upon observation of the pictures ensued:

Brenda: It's real close.

William: I think two-thirds.

Mathew: Look. Yeah, one and two-thirds is more. The boys get more.

All: Yeah, the boys get more.

A review of the procedure was reported to answer the question "How did you figure this out?" (e.g., "Give each boy one and cut the remainder in thirds. Give them two-thirds. Give each girl one and cut the remainder in fifths. Give them one-fifth more").

Summary: Language and Strategies Used to Solve Mathematical Problem Seven.

1. As a group, the children talked through the meaning of each part of the problem, and planned and labelled the solution prior to using any partitioning strategies. Having partitioned the circular areas, the group then determined three-fifths was more than one-third by referring to and discussing the illustrations.

Pizza Problem Eight

To share five pizzas among four boys and six pizzas among five girls, the group demonstrated the following specific strategies.

I. Understanding the Problem and Planning the Solution. To understand the problem, each child monitored the detail of Part A (the boys). The plan was stated by Mathew (e.g., "Give each boy one and cut the other one in four") and justified by Brenda (e.g., "Cause there's four boys"). Attention was directed to the girls by William (e.g., "Now the girls"), the solution proposed simultaneously by each child (e.g., "Give each girl one and cut the left over in five") and agreement reported (e.g., "They each get one and one-fifth"). By William raising a question (e.g., "They do?"), the solution was clarified by Mathew who quickly illustrated the solution and commented (e.g., "Yup. See!") and the solution was agreed upon (e.g., "Right").

II. Solving the Problem. Part A (the boys): As the plan was determined, the solution was achieved using one to one correspondence followed by partitioning in fourths.

This same procedure was followed to determine the solution to Part B (the girls) but the remaining pizza was partitioned in fifths using five radii.

III. Reviewing the Solution. To review the solution, each question was read and answered aloud in sequential order. No elaboration was evident.

Summary: Language and Strategies Used to Solve Mathematical Problem Eight.

1. As a group, the children talked through the plan and labelled the solutions prior to using any partitioning strategies.

2. The group recognized one and one-quarter was more than one and one-fifth by referring to their diagrammed solutions.

KNOWLEDGE OF MATHEMATICAL LABELS

A summary of the known or derived mathematical labels used by children in grades two, three and four while solving the pizza problems is provided (Table 4.1).

Mathematical Labels Used by Grade Two Children

The grade two children, individual and group, used only the specific mathematical labels one whole, one half and one quarter. Neither the individual nor the group used the term fourths. However, the group did recognize one half to be the equivalent of two quarters. Terms such as 'pieces' or 'shares' were used to identify the solutions for which the correct label was not known.

Mathematical Labels Used by Grade Three Children

The grade three children knew and correctly used the mathematical labels one whole, one half and one quarter. The individual derived and correctly labelled both partitions one third and two thirds. Two quarters and one half were used interchangeably by the group.

Table 4.1

Specific Mathematical Labels Used by Children in Grades Two, Three and Four While Solving the Partitioning Problems

Grade		Label											
		One whole	1/2	1/4	2/4	1/3	2/3	1 1/2	4/7	8/5	1 3/5	5/3	1 2/3
2	Individual	✓	✓	✓									
	Group	✓	✓	✓									
3	Individual	✓	✓	✓	✓	(✓)	(✓)						
	Group	✓	✓	✓	✓	✓	(✓)						
4	Individual	✓	✓	✓	✓								
	Group	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓

✓ - indicates the label was known.
(✓) - indicates the label was derived.

The grade three group, unlike the grade two group, knew and correctly used the term one third and an apparent understanding of the concept one third enabled the group not only to correctly derive but also to apply appropriately the mathematical label two thirds. When unknown, labels required to identify partitions of less than one third were simply called "quarters" or "pieces." The grade three individual child, however, endeavored to derive appropriate labels for partitions of less than one third through use of the label "quarter"; for example, a "quarter of a quarter" to mean one fourteenth.

Mathematical Labels Used by Grade Four Children

The grade four individual child knew the mathematical labels one whole, one half and one quarter. The child recognized that differing partitions do have specific labels but was unable to recall or derive the terms and referred to such partitions as "pieces."

Unlike the grade four individual, the grade four group knew all mathematical labels inherent in the solutions to the problems. The oral language specific to each problem indicated equivalent terms such as 8/5 or 1 3/5 could be used interchangeably. This concept appeared to arise as the children reviewed the solutions to each problem.

As might be expected, the specific use of mathematical labels available was greatest at the grade four level.

NON VERBAL BEHAVIOR OBSERVED WHILE CHILDREN SOLVED THE PARTITIONING PROBLEMS

Whether reading aloud or responding to the illustrations, each grade two child demonstrated similar sing-song, tapping, rhythmic motions while initially trying to understand the problem. Moreover, while planning the solution to each of the problems, both the individual child and the group were again observed demonstrating the same sing-song, rhythmic behaviors.

Similarly for all but the first and simplest problem, the grade three children were also observed manifesting sing-song and rhythmic behaviors; for example, counting in sing-song, body rocking, hair twisting, and pencil and finger tapping. Likewise, when planning solutions to the problem, the grade three children manifested the rhythmic behaviors previously described.

As the grade four group sought to plan the solutions for each of the more difficult problems, 4-5, the sing-song, rocking, rhythmic behaviors were demonstrated. Unlike the other children, the individual grade four child manifested no such sing-song, rhythmic behaviors as she silently struggled to understand and solve each problem.

It appears that the rocking, rhythmic movements were basic to the solving of the mathematical partitioning problems.

CONCLUSION

Within each step of Polya's problem solving framework there appears to be specific language functions and thinking strategies evidenced in the oral language used by the children. Oral language played an important role in the solving of the mathematical problems in that it allowed the young problem solvers to make a concerted effort to direct attention and clarify perception.

Chapter V

SUMMARY OF FINDINGS AND CONCLUSIONS, REFLECTION AND RECOMMENDATIONS

SUMMARY OF FINDINGS

The function of language used by young children, individually and in small groups, involved in mathematical problem solving, specifically partitioning tasks, was the focus of the present study. The focus was oriented towards the discovery of insights regarding the nature of the role of the language manifested and the thinking strategies revealed within the language while working on partitioning tasks.

Although a concerted effort was made by the researcher to be minimally involved in the interaction with the children, by design the study necessitated researcher involvement. By means of observation, questioning and ongoing analysis, it was possible to gain insight into how the children in this study used language to make meaning from posited partitioning problems. As well, the researcher was able to gain some insight into their partitioning behaviors.

Eight partitioning problems of graduated difficulty were adapted from Kieren's written mathematics tasks (1980) for the study, each permitting modifiable solution procedures.

No 'a priori' plans for analyzing subjects' responses were made. Instead, criteria for classification of subjects' responses were developed from the subjects' responses and in combination with relevant theory found in the literature. The reliability of the post hoc classification scheme was assessed by using one independent judge.

In this chapter the sample, the tasks, and the major findings and conclusions are summarized. Major reflections of the researcher regarding the learning behaviors of young children precede the recommendations for further research.

The Sample

The children were selected from classes in an elementary-junior high school within the St. Albert Separate School System. Six participating teachers, chosen by the principal subsequent to the explanation of the purpose of the study and the research methodology, selected a total of 26 children drawn from two classes of each grade, grades two through four. The total sample consisted of 14 girls and 12 boys and included 8 grade two, 8 grade three and 10 grade four students. Within each grade children were then randomly selected to compose two groups of three students. The remaining 14 children participated individually.

The Tasks

The partitioning tasks adapted and used displayed characteristics considered necessary for the purpose of the study. These characteristics were:

1. The problem solving task, sharing the pizza fairly, was believed to be familiar, motivational and conducive to generating language by young learners.

2. Varying mathematical aspects of the task would enable the children to experience success and, at the same time, challenge their thinking.

c. The gradation of the tasks would provide initial success that would motivate the children to move directly to partitioning.

FINDINGS AND CONCLUSIONS

Within the limitations of the study, conclusions have been drawn regarding the role of language used by young children when solving mathematical partitioning problems. The major findings and conclusions are stated in relation to the research questions posited and within the four problem solving steps of Polya.

Research Question 1

Do children individually and in small groups verbalize when problem solving?

In this study, individual children and those in small groups did verbalize while problem solving. The verbalizations were task oriented and extensive. Only the grade four individual child did not use language spontaneously throughout the problem solving process. This child, however, did respond to the probing of the researcher and eventually engaged in self-talk during the problem solving process.

Research Question 2

If children do use language, how and for what purpose is the language used, relative to solving the mathematical problem, individually and in small group settings?

Grade Two: The Individual Child

Understanding the problem. The individual child solved six problems. At the grade two level, the major language strategies used by the individual child were first to read the questions aloud to the self and then to the researcher. Having solved problems 1 and 2, the child thereafter either read silently or proceeded directly to the illustration. Language served to direct attention and to report recognized solutions.

Planning the solution. Throughout the planning stage, language served chiefly to direct attention via monitoring detail and activity and as well to predict possible alternatives and state conclusions.

Solving the problem. While solving each of the six problems, the functions of the child's language were to direct and to report achievement, analysis and conclusions.

Reviewing the solution. The review of each of the six problems evidenced the child mainly used language to report.

Grade Two: Group

Understanding the problem. To understand each of the four problems solved, the group members simultaneously read aloud, first to the self and then to the group. In addition, language was used to direct, to predict and to report.

Planning the solution. Throughout the planning the children used language to direct, predict and report. The thinking strategies apparent were analysis, synthesis, evaluation and justification.

Solving the problem. While solving the problems, the major purposes of the language were to direct and to report.

Reviewing the solution. To review the solution to each problem, the group read and answered each question aloud in sequential order. The review was used as an opportunity to debate, clarify and seek agreement.

Grade Three: The Individual

Understanding the problem. The individual solved six problems. Language served to report recognized solutions, to direct and to predict. Use of oral language was not evident when the child sought to understand problems 5 and 6.

Planning the solution. Having recognized the solution to problem 1, it seemed unnecessary for the child to plan a solution. As well, the plan to solve problem 2 appeared to emerge while making sense of the problem. To plan the solution to each of the remaining problems, the child used language chiefly to direct and to report.

Solving the problem. When solving problem 1, the child did not verbalize. In subsequent problems 2 through 6, the function of the language was to direct and to report.

Reviewing the solution. To review each of problem 1 through 6, the child read each question aloud and reported the answer. Embedded within the reports are efforts to: reflect upon meaning, justify, synthesize and draw conclusions.

Grade Three: The Group

Understanding the problem. Four problems were solved by the grade three group. To solve the first problem, the group began by simultaneously reading aloud. In subsequent problems, language was used chiefly to direct and report and to a lesser extent to predict.

Planning the solution. In problems 1 through 4, the language served to direct and to report. Extensive monitoring of detail was evident.

Reviewing the solution. To review, the grade three group used language primarily to read, direct and report. The reports served as a means to synthesize content, analyze and justify procedure, and to evaluate.

Grade Four: The Individual

Understanding the problem. The individual child eventually solved problems 1 through 4. Questions were read silently for each problem as it was presented. Language was used to direct attention to detail and simplistic reports were made which related to personal feelings.

Planning the solution. The plan to solve the first problem was directed by the researcher who used a series of questions. In subsequent problems, the language of the child served to direct and to report.

Solving the problem. Little or no talk occurred in this stage of the problem solving process.

Reviewing the solution. The review consisted of a series of reports which simply stated solutions but revealed few higher thinking processes.

Grade Four: The Group

Understanding the problem. The grade four group solved all eight problems and consistently first referred to the illustration to make sense of the problem. As the problems became more complex, problems 4 through 8, the use of language to direct increased. For this group, problem 4 appeared to be the most difficult. The thinking strategies apparent in the efforts to solve problem 4 were to question, justify, predict, and to state relationships.

Planning the solution. The solving of the first two problems had no stated plan. The plan to solve problem 3 emerged as the group related detail. In the remaining problems, the plan evolved as the children sought understanding by use of language to direct and report.

Solving the problem. Principal functions of the language used were to direct the activity and to report achievements. The role of language was minor in this problem solving step.

Reviewing the problem. To review each problem, 1 through 8, the group read and answered each question aloud. Terse reports served to explain and elaborate mathematical principles.

Research Question 3

Is there any difference between the language and strategies used by an individual and a small group of mathematical problem solvers at each grade level as well as across the grades two, three and four?

Since there are two aspects to Research Question 3, the summary of the findings is presented in two parts, that is, the performance of the individual versus the group (1) at each grade level and (2) across the grades.

Individual Versus Group

Grade two. At the grade two level, the individual achieved solutions for each of the first six problems and the group for only the first four. Both initially used the same problem solving procedures, that is, reading aloud to the self and then to another. Both used language to direct and to report when planning. The group, however, was much more verbal when solving the problem. The individual child relied upon the researcher to direct the review but the group independently reviewed the problem.

Grade three. To understand the problem, the individual child referred first to the illustration and then used language to report. The group either first read the questions then used language to direct, or immediately used language to direct. To plan the solution, the individual child used language to report and to direct. The group used similar strategies but the reports more often included analysis, justification, questions and evaluation. While solving the problem, both the individual and the group used language primarily to monitor ongoing activity. The individual and group reviewed the problems independently by reading and answering task questions aloud. The groups, however, more thoroughly endeavored in the reports to synthesize, justify, analyze, evaluate and conclude.

Grade four. The individual grade four child was guided throughout the problem solving process by the researcher. Hence, a conclusion regarding the role of the child's language in the problem solving process is difficult to establish. For this child, language served to respond in a series of simplistic reports.

To understand the problem, the grade four group initially read the questions to the first problem and thereafter made immediate reference to the illustration. Understanding the problem and planning the solution to each often appear to merge. Embedded in these two steps is the use of language to direct and to report. When solving the problem, little oral language was apparent. The grade four group consistently reviewed by having one child read each question aloud and all answered the questions.

Across the Grades

Understanding the problem. The grade two individual and group began the problem solving process by reading the questions aloud to the self and then rereading to another.

The grade three individual child read the questions aloud only once and thereafter referred immediately to the illustration in order to understand the problem. The group also immediately read the questions to the first problem aloud, and then reread the questions aloud several times to each other. Thereafter the group referred first to the illustration.

The grade four individual child silently read the questions to the first problem, reported difficulty and was led through the problem solving process by the researcher.

The grade four group read aloud the questions posed in problem 1, but in subsequent problems referred immediately to the illustration.

Planning the solution. The grade two individual used language primarily to question, direct and predict when planning a solution. Prediction seemed based upon past experience, hence trial and error prevailed. The grade two group used language to direct, report and predict. Prediction seemed to arise from past experience.

Understanding of problems 1 and 2 by the grade three individual seemed to be so clear that stating a plan was unnecessary. The language used in subsequent problems served to direct and report.

The language used by the grade three group functioned to direct and report as the plan unfolded.

The grade four individual child was guided by the researcher when planning the solution.

The plan to most problems emerged as the grade four group clarified their understanding. The function of the language was to direct and report detail.

Solving the problem. The individual grade two child used language to direct the activity and to report achievement when solving the problem.

The grade two group similarly used language to direct and as well as to report.

The grade three child did not verbalize when solving the first problem. Subsequent language used when solving the problems was chiefly to direct and to report.

Like the individual child, the grade three group used language chiefly to direct and to report.

The grade four individual child engaged in little or no talk throughout the problem solving step.

The grade four group also used little language while solving the problems. The language evidenced was to direct and to report achievements.

Major Conclusions

It would appear that the overriding function of language is to help the learner attempt to clarify meaning and bring sense to the experience. Not only did the language strategies shift within the steps of the problem solving process, they also seemed to vary in relation to age or stage of development of the problem solver across the grades. For example, the grade two children initiated problem solving by reading aloud. The grade three individual studied the illustration before silent reading but the group sought understanding by reading aloud. The grade four individual read silently but the group silently read the questions to the first problem and thereafter sought understanding by referring to the illustration. Furthermore, the grade two children sought solutions by talking through processes based upon trial and error. The grade threes talked through meaning based upon detail even when not able to demonstrate the skill. The grade four group immediately related detail and were able to demonstrate the skill. It seems unlikely that the grade two individual child would have reviewed the solution without guidance from the researcher. The grade threes semi-independently reviewed the solution and the grade four group systematically and independently reviewed each solution by first reading each question, then reporting and clarifying each answer. Clearly, the incidence of talk increased as the complexity of the problem intensified; inversely, the talk dimished as the grade level increases.

Research Question 4

> *Is there any developmental pattern of growth and change in the mathematical problem solving procedures of children in grades two through four?*

There appear to be distinct levels of partitioning capabilities with each level, characterized by developing concepts and processes. For example, even when achieving the solution, the grade two children employed a process of partitioning, much like dealing cards. The procedure is first to group, and then to find one-to-one correspondence. To partition a circular area into thirds tended to be an arduous task, even when talked through and the children were sometimes not able to execute the task despite being able to verbalize clearly what must be done to demonstrate the solution. The grade three children seemed able to perceive partitioning from a broader view and rely less upon the process of 'group and deal.' They also were able, for example, to distinguish two-thirds as part of one whole in contrast to the grade two strategy of collecting thirds and adding.

Grade four children (group) examined and related details inherent in the problem and quickly determined the partition.

Research Question 5

> *What is the role of the teacher relative to the problem solving process?*

As the focus of this study is upon how children come to know the researcher was able to glean insight and speculate regarding the role of teacher in the problem solving process.

It has been reported that children do not easily bridge the gap between problem settings and their symbolic representation (Resnick, 1982). Further evidence substantiates the fact that children have their own ways of constructing meaning of problematic situations (Carpenter and Moser, 1982). It is the role of teacher to design a strategy which would address these concerns. Such a strategy necessarily requires that the teacher recognize the meaning vested within the concepts, partitioning, computing or other, must inform the learning that will eventuate in them (Sawada, 1984). Further, problem solving can serve as the integrating medium of concrete-manipulative representation and symbolic insight. The environment within which the learning experiences must be contexted within the school is the socio-linguistic environment within which the children live outside the school. It is the responsibility of the teacher to design problem solving tasks which are meaningful to the child. Teachers should pose specific problems based upon real life experience.

The problem, itself, must be inherently interesting to maintain pupil attention. Problems too narrowly defined allow little opportunity for children to explore and create. Conversely, problems too broadly defined lessen the likelihood of resolution.

As well, it is the responsibility of the teacher to provide opportunity for children to work together or alone in a true problem solving environment in order to access some of the children's personal knowing and to provide the child as knower opportunity to understand, to make sense of, and to come to know the meaning of the manipulative and symbolic representations with which they learn.

This view places the child at the centre of the problem solving experience. This is not to imply that pupil talk is superior to teacher talk, that pupil direction is superior to teacher direction. Surely this is not so as the teacher's role is a vital one. Nonetheless, implications for instruction are apparent.

Small groups efficiently serve the problem solving process as a strategy to be used to promote exploration of a concept, attitude or skill. The use of this strategy at the right time provides opportunity to enable the learner to come to know. Thinking is stimulated in the group situation as is clarification of thinking. Subsequently, problem solving is enhanced by group interactive processes but whether in groups or individually the problem solving process is neither quiet nor still.

The teacher should also carefully plan the physical organization for children to engage in small group problem solving situations. The organization must provide a face-to-face situation in order that comments can be directed to the group and that group members can react to the ideas being offered or the activity being pursued.

Typically, teachers 'talk and chalk,' demonstrate and show the learners what is to be learned. This role of the knowing authority needs to be altered to provide unobtrusive assistance that will facilitate the individual or group problem solving process. The teacher's purposes are three-fold: to listen, to question, and to know when to terminate. Listening accompanied with careful observation enables the teacher to determine how children use their language to solve problems within varied contexts. Having listened, teachers are in a better position to encourage children to extend their experience, to explore reasons for ideas, to exercise critical thinking and to learn from each other. Questioning techniques provide opportunity to correct misinterpretation or misinformation, to open a new line of thought, to probe and encourage deeper development of an idea, and to involve the more timid child as a group participant. In some instances, it may also be the responsibility of the teacher to recognize when the problem has been 'talked out' and to bring the activity to a satisfactory conclusion.

It is extremely important that the classroom teacher carefully prepare the problems to be resolved and the kinds of problem solving experience to be offered. Consideration of the most propitious conditions in which to offer problem solving experiences is a major responsibility of the teacher.

EPILOGUE: REFLECTIONS OF THE RESEARCHER

Any study of the uses of language for learning is of more than theoretical interest. An examination of the role of language in problem solving, mathematical or otherwise, can not preclude retrospective reflection upon several pertinent questions. The most important question arising from the learnings of this study is, 'How best can the classroom teacher provide opportunity for learners to learn, to use and strengthen problem solving skills.' This question necessarily requires that several issues be addressed. First, what are the attributes of a problem and what constitutes problem solving? Second, although it would be presumptuous to equate language or speech with thought, oral language does seem to play an important role in the cognitive process. Can language serve the learner engaged in a problematic situation, and if so how does it serve him? Third, can and how does interaction serve the learner, hence the problem solving process?

Problem Solving

What constitutes a problem, mathematical or otherwise, for a young child? A problem, by definition, exists when an individual or group encounters a situation which requires resolution and for which there is no readily available solution. Problems posed within the classroom must arise from the student's own real world interests. Moreover, there must exist for the child the possibility of achieving a real and applicable solution. Simply stated, the problem posed must enable the child to make sense of the situation.

Language and Problem Solving

In one sense, thinking is regarded as a process of using ability, ofttimes intelligence, to solve problems. This limited definition does not necessarily include the concept of understanding or clarifying a situation. The derivation of the term clarify is from the Latin words clarus and facere, meaning 'clear' and 'to make'; hence, 'to make clear.' By definition, clarification is to make an idea, statement, clear or intelligible; to free the mind from ambiguity or confusion. In the broader sense, thinking then, is a process directed to explore, clarify and enlarge experience. Thinking is then a deliberate exploration of experience for a purpose. It is the exploration which clarifies perceptions and enables one to arrive at a judgement, i.e., to resolve the problem.

The relationship between the use of oral language and thinking is, therefore, an important one because language is the code system a child develops for exploring experience. By exploring experience, gleaning perception, asking questions and making assumptions, however elementary, the child obtains information. Thinking is concerned with extracting information from experience. For the child, oral language is a natural process for extracting information, and therefore, to think.

Emphasis upon verbal expression is important. However, verbage by itself is inadequate and necessitates examination of the language skill to thereby determine the underlying thinking strategies. The job of thinking is to clarify perceptions in order to have a clearer view; a process to direct attention and clarify understanding. The children in this study used language to a great extent for exactly that purpose, to direct attention over available knowledge, to clarify and validate perception.

Perception of a mathematical problem is a matter of awareness and that is one of the functions of thinking, to clarify awareness, to free from indistinctness or ambiguity. To clarify is an overriding function of language. Higher level mathematics is used in order to 'see clearly' what is implicit in a set of given relationships. In the mathematical procedure, the situation is processed in order to perceive the mathematical relationships more clearly. Typically, the mathematical problems given children require more ordinary thinking and the child can do much to explore the experience, to direct attention before moving into the processing stage. Oral language is key to the exploration and clarification of not only the problem, but also the ongoing problem solving activities.

To know is to perceive or understand, to apprehend clearly and with certainty, to establish or fix in the mind. Oral language is a means which enables the child to come to know. Language not only provides the child with useable concepts and a means with which to clarify the world but is also the system a child develops for dealing with other people.

Interaction and Coming to Know

There appears to exist a pervasive relationship between learning and its context. It could be said that children's capacity to talk is there, just as the capacity to walk or see, and all that educators need do is to draw upon this capacity but this view falls short of the goal. What is compelling is the realization that children learn through talk and the way they learn is complex and varied. Nonetheless, it appears that small group interaction is highly conducive to the learning process.

Observation of small groups interacting to solve partitioning problems permits conclusions pertaining to the process of interaction.

The first observation and most obvious conclusion is that talk, though rich in meaning, is untidy and appears fragmented. Consequently, group deliberation is a slower method to arrive at a solution. Nonetheless, it appears to be immensely satisfying.

Second, problem solving is an attempt to transform uncertainty into familiarity (Smith, 1975). Oral language serves the child as a means to construct meaningful patterns relating what is understood to what children need to understand. As the learners interact with their surroundings, tasks, peers and activities, concepts are enlarged, experiences re-shaped and insights gained. Language marshalls forces of attention, serving to attend and help execute the task. Interaction motivates to put impressions in words, thus making impressions more precise. The struggle to re-present understandings in spoken form gives ideas shapes.

Given opportunity to talk, even young children seem able to manipulate concepts in the absence of a concrete referent. Children predict, speculate and reason by means of language thereby moving beyond the immediate to new arrangements and possibilities. The language of children interacting makes possible control over comprehension and learning aiding the move to a more variable view.

Third, the problem solving process begins with identification of the problem, which, for the child, is often the most difficult step. Once the problem is identified, for older children, in particular, the solution often appears to be obvious. Younger children, seven and eight year olds, search for solutions based upon trial and error, the referent being past experience, but the identification and the planning stages appear to be highly verbal.

Fourth, activity, rhythmic behaviors, movement are a natural part of learning. The sitting-down child is surely a highly unusual phenomenon in real life experiences and also such in the learning experience. When problem solving, a tension appears to be created and, with the tension, children become more active but not less attentive.

Fifth, small groups provide a supportive context for learning. Feedback is prompt, signalling success or failure. Disagreement, contrary to being defeative, spurs further activity and motivates. There appears to be strong cohesiveness and morale as well as commitment to the task.

Sixth, an almost uncanny phenomenon appears to exist which permits the members of the group to be privy to information unknown to those outside the group. Unstated assumptions of the group members seem to determine the interpretation of the experience. It is as though the participants agree on how the talking will be done and how it is to be interpreted. The agreements seem momentary and the children understand each other because, for the while, there exists an assumed reasonableness of each other's statements. Out of fragmentary data, reasonableness is constructed. Learning can be likened to a ping-pong ball where the move of each player is to some extent dictated by the previous moves of the other players. The children are active rather than passive players in their own learning.

In summation, interaction enables children to pursue a problem more deeply, to ask questions, to try out ideas, and to finish working in a caring way.

RECOMMENDATIONS FOR FURTHER RESEARCH

The findings of this study have generated the following suggestions for further research.

1. Observation: The present investigation presented tasks at the symbolic level.

Research Question: Would manipulative materials evoke similar language/thinking strategies and behaviors of young children while involved in partitioning tasks?

2. Observation: In this study, the mathematical problems posed were partitioning problems.

Research Question: What would be the function of the language used by young children engaged in other mathematical problems?

3. Observation: This study, concerned with the language/thinking strategies demonstrated by young children engaged in mathematical partitioning problem solving evidenced little use of imaginary language.

Research Question: To what extent are the language functions determined by the pupils' perceptions of the context for that language and the kinds and levels of thinking required to solve problems?

4. Observation: The overriding function of the language used by children in this investigation was an endeavor to clarify thought throughout the problem solving process.

Research Question: By replication of the study, could more conclusive information be derived from which generalizations could be made regarding developmental patterns concerned with growth in language/thinking strategies used in mathematical problem solving?

5. Observation: In the present study, the lower the grade level, the greater the incidence of oral language generated. The apparent need to verbalize tended to diminish as the grade level increased.

Research Question: Does the clarifying function of language diminish as the grade level increases or is the unprompted use of oral language inverse to the ease with which the solution to the problem is secured?

6. Observation: Students at the grade two level probably would have neglected to review the solution to the problem without the intervention of the researcher. It seems unlikely that young students perceive the process of problem solving as an end in itself enabling them to learn the generalizations which will transfer ability to solve other problems.

Research Question: Would discussion and oral study of the problem solving process directly evidence a transfer of training with significant success to new and unfamiliar types of mathematical problems?

7. Observation: In this study the children talked spontaneously throughout the problem solving process.

Research Question: How does the verbalization of a problem or concept affect the way it is understood or learned?

8. Observation: In this study, the children were persistent and successful in their efforts to unravel meaning, leading to subsequent solution of the posed problems.

Research Question: What concepts, skills and predisposition are associated with success and pleasure in the learning and solving of partitioning problems.

9. Observation: Rhythmic behaviors and sing-song were manifested by individual children and small groups when seeking to understand and plan the solutions to the posed problems.

Research Question: Is this behavior characteristic of young children exposed to problematic situations?

10. Observation: Though able to verbalize and reflect upon the necessary procedure to achieve partitioning in thirds, the task was arduous for the grade two children.

Research Question: Is the inability to partition in thirds a perceptual problem characteristic of seven year old children?

11. Observation: A knowledge of labels seemed evident when the task appeared to be relatively simplistic.

Research Question: To what extent does the awareness of an appropriate label aid the recall of experience to contribute to the successful solution of a problem?

12. Observation: The enhancement of problem solving by group interaction and group effect is very intriguing.

Research Question: To what extent is the development of team building influenced by prompting a sense of community? Furthermore, to what extent does a common purpose influence learning and problem solving?

CONCLUDING STATEMENT

Basic skills of thought, making meaningful messages and making sense of messages from others, are developed at the levels of experience and oral speech, i.e., within the socio-linguistic culture of the child. The matching of thought with speech and of the individual's own mind with the minds of others, through oral language, enables a child to make sense of the experience. Oral language serves to clarify problems as well as to find solutions by gradually enabling a child to access personal knowing, in order to come to know and to discover meaning for himself. Without the basic cognitive skills—making messages and making sense of messages—the position of the learner is, at best, perilous. Otherwise, how can he advance toward the extended symbolic schemes of mathematics?

In the context of problem solving, the role of oral language is crucial. It serves as a powerful means to aid children to build their own connections between concrete-manipulative experience and the symbolic reality of mathematics.

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APPENDICES

APPENDIX A

ORIGINAL TASKS

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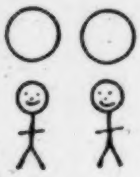

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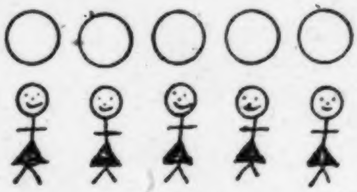
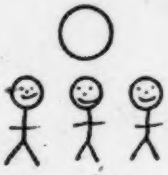


B

Who gets more? A Same B

How did you figure this out?

2.



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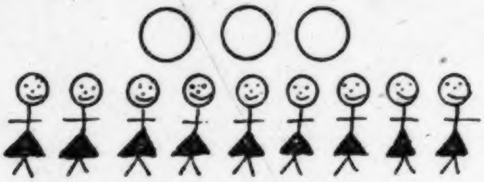

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Who gets more? A Same B

How did you figure this out?

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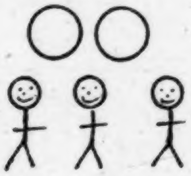

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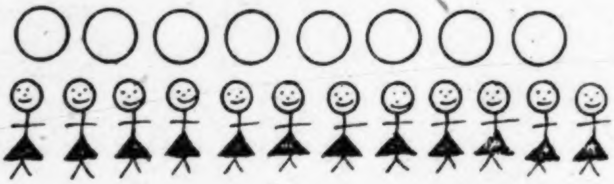

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Who gets more? A Same B

How did you figure this out?

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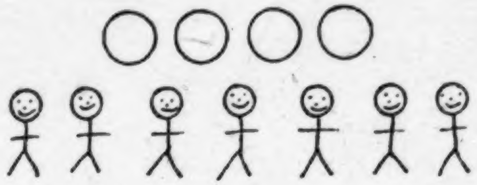

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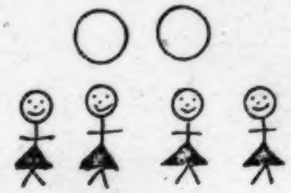
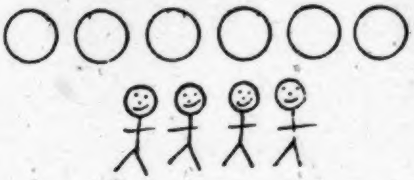


B

Who gets more pizza? A Same B

How did you figure this out?

5.



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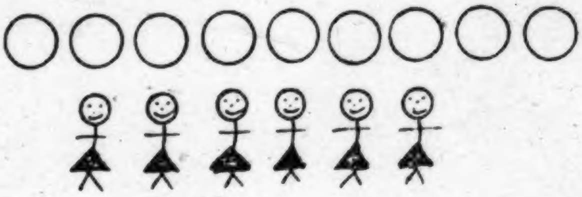
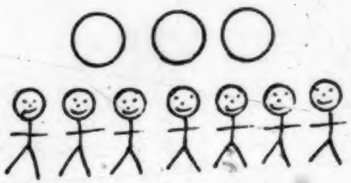


B

Who gets more pizza? A Same B

How did you figure this out?

6.



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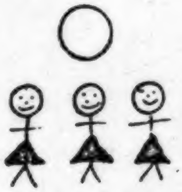
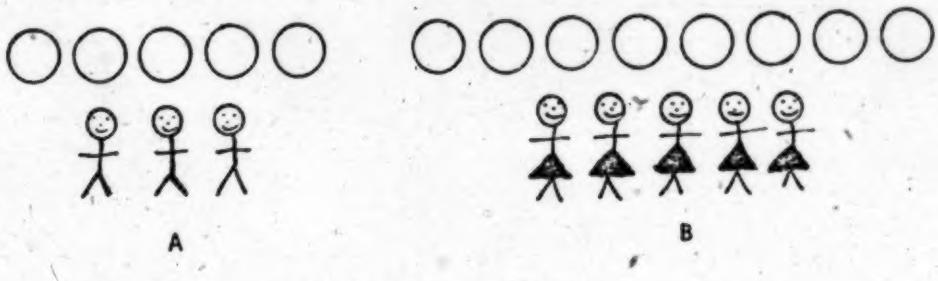


B

Who gets more pizza? A Same B

How did you figure this out?

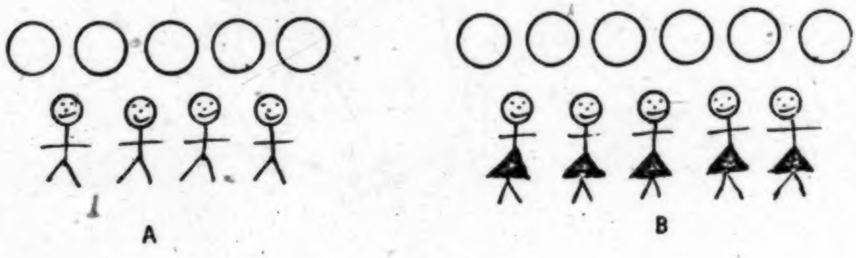
7.



Who gets more pizza? A Same B

How did you figure this out?

8.



Who gets more pizza? A Same B

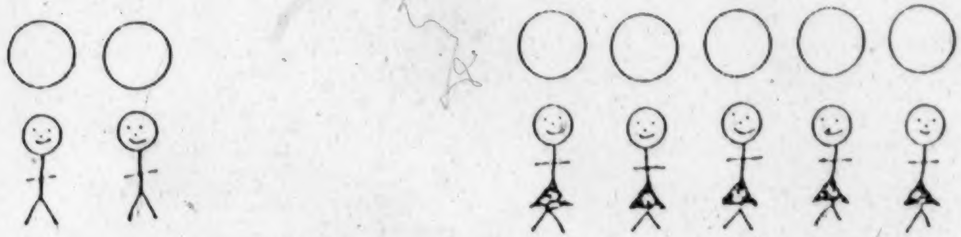
How did you figure this out?

APPENDIX B

ADAPTED TASKS

SHARING THE PIZZA

1.



How much pizza does each boy get?

How much pizza does each girl get?

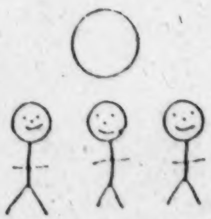
If each child gets a fair share of the pizza, who would get more pizza - - - each boy or each girl?

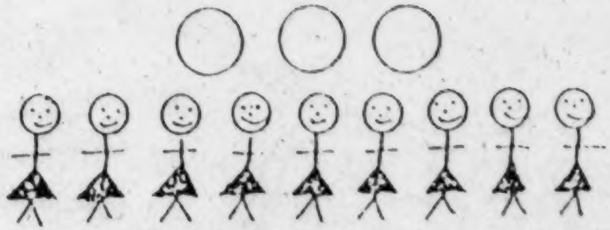
Does each boy get as much pizza as each girl?

How did you figure this out?

Answer below.

2.





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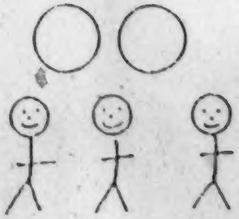

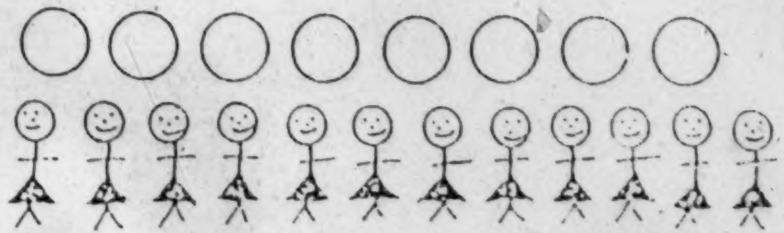

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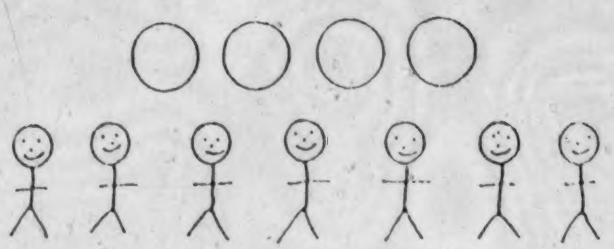

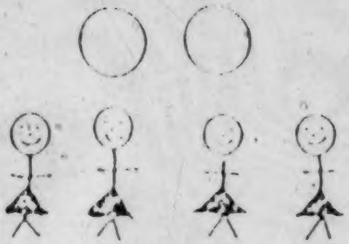

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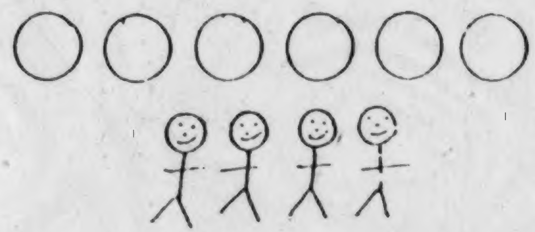

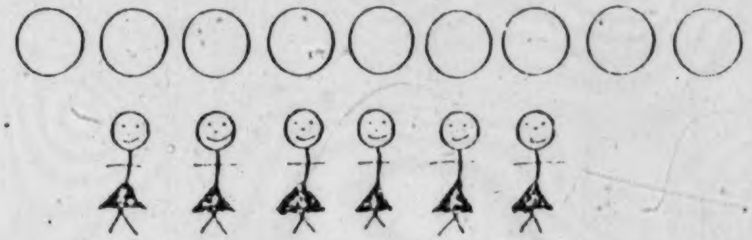

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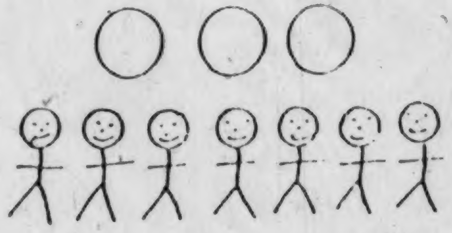

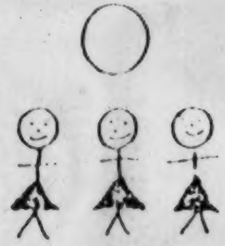

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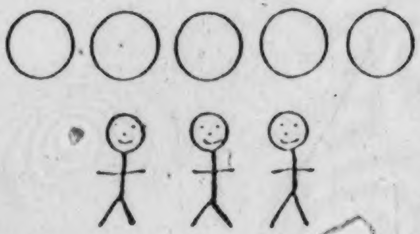

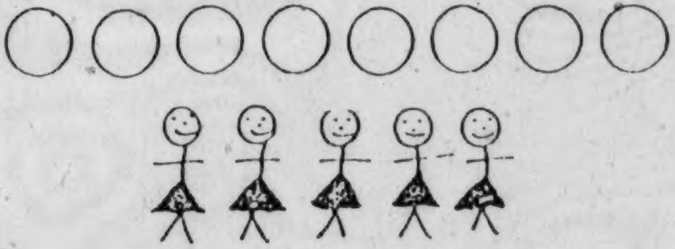

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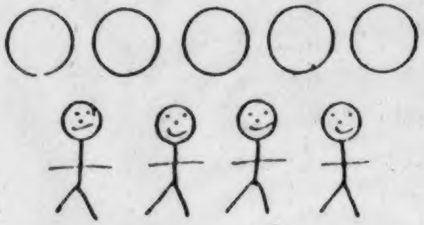
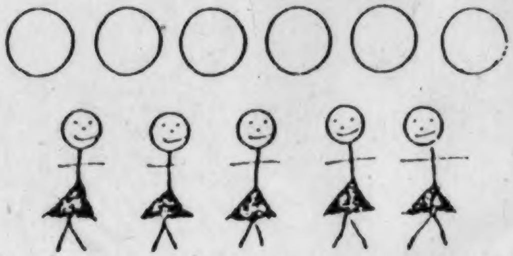




APPENDIX C

LETTER TO THE PARENTS

Elmer S. Gish School
75 Akins Drive
St. Albert. Alberta
T8N 3B5

Phone: 459-7756

September 15, 1982

Dear Parents:

This fall I will be interviewing children at E. S. Gish School as a combined enrichment - research project in language/mathematics education. I would like to videotape the interview sessions as the best method of recording the data for subsequent analysis. Whenever the tapes will be viewed the identity of the children will always be withheld. Also, any written analysis of the data will conceal the child's identity.

I have the permission of both Mr. J. Bauman, Principal of E. S. Gish School, and Mrs. M. Martin, Assistant Superintendent of the St. Albert Protestant Separate School District No. 6, to conduct this combined teaching - research study.

If you have no objection to your child's participation being recorded in thesis format, would you kindly fill in the attached form and have it returned to your child's teacher.

Sincerely yours,

Beryl Wales

Beryl Wales (Miss)

BW/sls

St. Albert Protestant Separate
School District No. 6

PLEASE DETACH AND RETURN TO YOUR CHILD'S TEACHER

I am willing to have my child's participation in the research project in language/mathematics education documented.

NAME OF CHILD: ________________________________

DATE: ______________ PARENT'S SIGNATURE: ______________________

PLEASE DETACH AND RETURN TO YOUR CHILD'S TEACHER

I am willing to have my child's participation in the research project in language/mathematics education documented.

NAME OF CHILD: ________________________________

DATE: ______________ PARENT'S SIGNATURE: ______________________

PLEASE DETACH AND RETURN TO YOUR CHILD'S TEACHER

I am willing to have my child's participation in the research project in language/mathematics education documented.

NAME OF CHILD: ________________________________

DATE: ______________ PARENT'S SIGNATURE: ______________________

APPENDIX D

PROTOCOL: GRADE TWO, INDEPENDENT

GRADE TWO

JOHN, INDEPENDENT

Pizza Problem One

R This is pizza problem one and what I would like you to do is decide how much pizza each child will get if you share the pizza fairly. Notice these are the boys and their pizza and over here are the girls and their pizza.

J (Reads questions)
How much pizza does each boy get?
How much pizza does each girl get?
If each child gets a fair share of the pizza, who would get more pizza—each boy or each girl?
Does each boy get as much pizza as each girl?
Pizza -- right -- it's confusing. Well, I think so -- if it's fair shares.
How did you figure this out? Answer below.
(Looks at teacher and rereads question aloud.)
Hmmm. Do I just draw in the pieces here, to make fair shares?

R Maybe you'd better do the boys first.

J That's a question. He gets four pieces. (Draws) There.

R Can you tell me how much pizza each boy gets?

J Four quarters.

R What's another name for four quarters? How much has he got?

J (Pause) Four pieces -- that's one pizza.

R Now find out how much each girl gets.

J Okay. Just cut them like that. Okay. That's her pizza.

R Can you tell me why you're dividing them up?

J Yup. I'm dividing them up so each boy and girl get -- like -- four pieces. Cause if I was dividing them up into four over here and three over here it wouldn't be fair shares.

R That's true.

J Hmmm.

R How much pizza does each boy get?

J One whole piece -- actually one pizza.

R How much does each girl get?

J One pizza.

R If each child gets a fair share, who gets more, each boy or each girl?

J They get the same amount.

R Does each boy get as much as each girl?

J Yeah.

Pizza Problem Two

R This is pizza problem number two and what I would like you to do is decide how much pizza each child will get if you share the pizza fairly. Notice these are the boys and their pizza and over here are the girls and their pizza.

J (Reads questions)
How much pizza does each boy get?
How much pizza does each girl get?
If each child gets a fair share of the pizza, who would get more pizza—each boy or each girl?
How did you figure this out? Answer below.
(Looks at teacher and rereads question aloud.)
Hmmm. These are the same questions. Fair shares?

R It should be in fair shares, right. You have to consider fair shares.

J Make it equal to six pieces -- across and -- three boys -- no -- eraser --

R What are you trying to make?

J Pieces.

R Why?

J So each boy gets one fair piece of pizza.

R It won't bother you if I sit here and sort junk will it?

J No. (Whispering) Like that.

R Why did you divide it in threes?

J So each girl gets one piece of pizza so it will be even with the boys. I divided it into three.

R Okay. How much pizza does each boy get?

J One piece of pizza.

R Out of how much?

J Out of three pieces.

R Do you know a name for that. One out of three?

J One divided by 3 is 2.

R Okay. What about the girls? How much does each girl get?

J One pizza divided by 3.

R Do you know the name for that?

J One divided by 3 equals 2.

R Can you write 1 divided by 3?

J -- dividing sign?

R How do you think it might go? Any idea how. Take a guess. How do you think it might go?

J Hmmm.

T Where would you put the 1?

J This, 1 and 3 over here. ($\frac{1}{3}$)

R Can you look at that. Do you know a name for that?

J Not really.

R Okay, let's do the next question. And it says, who would get more pizza each boy or each girl?

J I think they get the same amount—cause each boy gets one piece of pizza out of three and each girl gets one piece of pizza out of three.

R So the next one says, does each boy get as much pizza as each girl?

J Yeah.

R How did you figure it out? Now, you don't need to write it. You just tell me.

J I figured it out by starting in the center and drawing three lines out from it.

R Why did you do that?

J So I can look through this straight and see if it's -- equal.

R How many pieces were you trying to get?

J Three.

R Why did you need three pieces?

J I needed three pieces cause there is three boys.

R Now tell me how you got the girls.

J I divided the girls up into groups, three groups, this pizza, this pizza, this pizza.

R Good. Then what? After you divided the girls, what did you do?

J I drew the lines so each girl got one piece.

R Out of?

J Out of three.

R So you had basically this didn't you. This --

J One, two, three.

R So you got the boys figured out and the girls figured out. Then what did you do? How could you tell they got the same?

J -- I took -- took them out of one pizza and put it with the other and the other.

R That was smart.

J Then one pizza is for -- each three.

R So then you looked at what the boys got.

J Umhumh.

R Then you looked at what the girls got and you decided they got the same. That's good. Okay. Put your name on that.

Pizza Problem Three

R You're pretty good at this. This one is a little tricker, not much but a little.

J -- cause this girl doesn't have a pizza. I don't want that one, this girl gets one whole pizza these only get a half.

R So, try to think of another way.

J This one is tricker. There's three for one here and there's three for two here -- might be easier -- this is really tough.

R But I'm sure you can do it.

J -- three boys, let's see. I put -- pizza in halves and half left. (Long silence)

R What else could you do?

J One pizza at a time, have to cut it into -- pieces --

R Then what would you do with the next pizza? What would you do with the second pizza?

J Give it to the girls.

R No, you can't do that, that's the boys.

J I think I'll try that -- that's two each. There -- and there. (Whispering)

R Did you get the boys? How did you do?

J The boys get two pieces.

R Okay. Now, let's see how you figure out the girls.

J Probably the same way -- She's for that pizza. She's for that pizza. She's for that pizza. -- That would be two girls for another pizza -- think it is one pizza -- eight pizzas -- four more pizzas --

R I'm going to leave you for a few minutes and just study while you can get that. I'll put my money on you -- if I give you time.

(Long silence)

J I don't know how -- where's that teacher?

R Well, how are you doing?

J Not too well.

R Tell me what you've done so far, or what you've tried to do.

J I was trying to -- these girls one pizza -- of girls --

R Two, three for one. Is that what you're saying?

J Ummhmm.

R One, two three, for one, two three for one, you're on the right track. I think I left one. How many pizzas left over?

J Four.

R How many groups of girls do you have?

J Four.

R So what could you do with the four pizzas that are left over?

J One more pizza for each group --

R Now you've got how many pizzas for each group?

J Four.

R You gave one to each group. If you give another one, how many pizzas is that for each group?

J (Silence)

R Okay. You had one and you said you gave them another one, so they would get how many?

J Two.

R -- so --

J How am I going to divide it?

R You're right on.

J Oooops.

R Well, how much pizza does each boy get?

J Two pieces of pizza.

R Out of how many?

J Three.

R Out of three pizzas?

J Out of six pieces.

R How much do you get out of each pizza?

J How much did you get out of EACH pizza?

R Umh.

J One piece.

R Out of?

J Six.

R Six pieces in one pizza?

J Well, three.

R Okay, so then tell me again how much does he get out of each pizza?

J He gets one piece out of three pieces of pizza.

R Okay. Now what about the girls? How much does each girl get?

J Two pieces out of six pieces.

R Or?

J One piece out of the big pieces.

R So if each child gets a fair share, who would get more, each boy or each girl?

J They would get the same amount.

R How do you know?

J Cause if it's fair shares, fair means you get the same amount.

R How much did each boy get?

J Two pieces.

R How much did each girl get?

J Two pieces. Then it says does each boy get as much pizza as each girl. Umhum.

R Then it says how did you figure it out. Now we have to think back about what you did. What's the first thing you did? When you started to get the answer, what did you do first, the boys or the girls?

J The boys.

R Then what did you do?

J I -- I cut this pizza into three pieces. Cause there was -- three boys.

R And then?

J I cut up three more pieces.

R How did you share them?

J I gave each boy two pieces.

R Now let's look at the girls 'cause that's what you did next. After you fooled around for awhile then you got on the right track. What did you do? Well, first of all, you divided up the girls, didn't you? Then you gave each group of girls --

J One pizza.

R -- and you had some left over. So what did you do with the left overs.

J I gave the left overs to the groups. Each group gets two pizzas.

R Then -- you were so close, what did you do next?

J I divided the next pizza up into three so each girl gets two pieces.

R Right -- and then -- How did you find out if the boys had as much as the girls?

J I took my fingernail and measured it.

R And you also saw that the boys got how many pieces?

J Two.

R And the girls got --

J Two.

R -- that's long enough -- Put your name on the page. Do you need a rest?

J No.

Pizza Problem Four

R Okay. By the time you're through you're going to be the best pizza cutter in the world.

J Oh, oh! Now look at all the boys.

R I'll just leave you to puzzle it out.

J -- some pretty tough. The girls are easy to divide. I know it. (Whispers) -- It wouldn't work. Two boys got one pizza cause there will be one left over. I'm thinking about two boys for one pizza but then there's one left over. One boy left over. And one pizza.

R So what would you do?

J What would I do? -- three pieces, no I have to --

R Remember it has to be fair shares.

J It can't be. One guy gets one whole pizza and one guy gets an itsy bitsy piece.

R Right.

J They would be a little hard cutting into seven pieces. (To self)

R What did you say?

J It would be a little hard cutting it into seven pieces again. (To self and looking at second pizza) -- might get two pieces.

R Let's see how you can do it, the best way you can do it.

J -- turn it around -- sure I could do it. Four -- half. I know. I'll cut it into five pieces -- 1, 2, 3, 4, 5 -- 6, 7. I've got seven pieces. What's my answer?

R Well, what's your first question?

J My first question -- get how much. Two pieces -- one piece -- I think.

R How big is each piece?

J That the girls get?

R Umhum.

J A half a pizza.

R Let's go back to the boys. How much did each boy get?

J Two pieces (inaudible) I think.

R Now you said that each child gets a fair share of the pizza. Who would get more, each boy or each girl?

J Each boy, two pieces.

R How much does he get? More or less? How would you figure it out?

J (Inaudible) Each girl would get a half a pizza.

R And each boy?

J Four out of seven.

R -- four -- right. So who gets more, each boy or each girl?

J (Inaudible) Let's see. (Looks at picture. Folds 4/7 over 1/2.) The boys.

Pizza Problem Five

R Here's problem number five and let's see how you try and do it.

J Two pizzas for two people and there would be two left over. So that might be three pizzas for two kids equals three pizzas for two kids. So I think they have to go that way.

R What about the girls?

J That's a little tougher. It's two for three. It has to be two for three. How can I divide into four pieces? -- get three pieces, so half right in the middle -- one, two -- there's two for three, two for three, and two for three. Each girl gets three pieces -- three pieces.

R Pretty quick. How much pizza does each boy get?

J (Reads the questions.) How much pizza does each girl get? If each child gets a fair share of the pizza, who would get more pizza each boy or each girl? If they got fair shares they would have the same amount.

R How much pizza does each boy get?

J In a group?

R Well, there's six pizzas. How much of the three pizzas goes together?

J How much does it go together? If you put them together that would make one pizza and a half.

R How much does each girl get?

J A pizza and a half. The two halves together and you would have one more half left over so it's three pieces -- It's a pizza and a half.

R Good. Then it says who gets more each boy or each girl?

J I think they get the same amount.

R How did you figure it out? Just tell me. You don't need to write it.

J Okay. I figured it because, first I tried two people for -- two boys for two pizzas. That didn't work. So I tried three boys for -- two boys for three pizzas and then they had -- I divided so -- so that each boy gets same amount. Then girls by three pizzas for two girls and then I drew in a line to divide the pizza in pieces. That's how I figured it out.

Pizza Problem Six

R Here's pizza problem six.

J Oh, oh! I know -- the girls aren't hard. Let's see if I can figure out the boys. Let's see. One for one pizza. Nope that didn't work. Two for one pizza, no. Three for one pizza. Oh, oh. I took my name -- seven boys and three pizzas -- give one pizza to each boy, there would be four boys left over.

R What happens if you try doing it the same way as the girls?

J Umh. So that would be -- divided like that, so if I divided it like that -- I think I know what would happen -- I'll see. If I divided this way -- these guys get it -- three -- these guys would get this. Then this guy would get that -- This one doesn't get near fair. He's not having a fair share. 'Cause these guys get a small piece and he gets a great big pizza. Cut it into seven pieces.

R -- What should he have?

J What he should have is something like that.

J I know, get seven more pieces -- cut this into seven pieces but then it won't be fair shares. Let's see -- umm.

R Can you tell me how much each boys gets?

J One piece.

R How big is that piece?

J How big is that piece?

R One piece out of how many?

J Three.

R Okay. How much does each girl get?

J One.

R Out of how many?

J Out of three.

R What are you going to do with that left over part?

J I don't know. (Pause) Cut it into seven more pieces! 1, 2, 3, 4, 5, 6, 7 -- more pieces have to go. Not big enough -- I divided the one piece left over.

R You took this piece out and then what did you do?

J What's left? The pieces that I cut out for the boys were seconds.

R What did you divide that into?

J Seven pieces.

R Why was that?

J There was seven boys here and each one of them got one piece and there still was some more left over so I decided to cut that in sevens. -- One more time, so they each get two pieces.

R Good. Then it's pretty easy to see who got more. Right. Who did you say got more?

J The boys!

Pizza Problem Seven

J Girls six -- It gets tougher all the time.

R You think it's pretty tough for you?

J Not really, but I can't wait to see what eight is. But I'll have to.

R See how you'll handle that one.

J I should.

R I think you'll do it.

J Should I give one pizza to each girl, there is two pizzas left over.

R Then what happens if you do that to the girls?

J I give one to each, the whole -- no there's three left over.

R Maybe a puzzler? Want to continue?

J Hum.

R Do you want to continue or do you want to stop?

J Eight, continue.

R Continue on seven.

J I guess so. Only way I'll get it done.

R You don't have to you know. I don't really expect you to get seven. Can you tell by looking who would get more? It's pretty tough isn't it.

J Hum. Yeah. I think I'll quit.

APPENDIX E

PROTOCOL: GRADE TWO, GROUP

GRADE TWO

GROUP: ANNIE, HELEN, KARLA

Pizza Problem One

R This is a problem about sharing the pizza. What I would like you to do is decide how much pizza each child will get if you share the pizza fairly. Notice these are the boys and their pizza and over here are the girls and their pizza.

R What are you doing?

H I'm putting on the pepperoni.

R Why is that?

H 'Cause these pizzas only have cheese.

R Oh.

K Do it later.

All (Read questions out loud to the self. Read questions out loud to each other.)

K Look it. 1, 2, 3, 4, 5, 6, 7. There's seven pizzas and there's 1, 2, 3, 4, 5, 6, 7—7 kids.

H 1, 2, 3, 4, 5, 6, 7. 1, 2, 3, 4, 5, 6, 7. Yeah.

K So they each get one.

All Yeah. They each get one!

K Do we put a ring around them?

R Whatever you want. You're the ones who are sharing the pizza.

A Put a ring around.

R What are you doing now?

H I'm putting it in his hand so he eats it nicely.

K Yeah. Cut it in pieces -- the same kind of pieces.

A Make sure they're the same. Cut them right.

H	There. I'm done.
K	There. I'm done.
A	Yours are different than mine.
K	Doesn't matter long as they're the same.
H	Oh.
A	Let's see. Now let's do the questions.
H & K	Yeah.
All	(Read aloud all of the questions in sequential order.)
K	Who writes the answer?
A	We all do. She gave us each a page.
H	How much pizza does each boy get? One.
A	How much pizza does each boy get? One. One whole pizza.
K	How much pizza does each boy get? One. One whole pizza. Now, How much pizza does each girl get? One.
H	One.
A	One whole pizza. That's a lot, you know.
K	Does each boy get as much pizza as each girl? Yeah. They each get one pizza.
A	Yes, 'cause it's fair shares.
H	Yes. One.
K	How did you figure this out? Answer below.
H	How did you figure this out? Answer below.
A	Do we write that down?
R	No, you can just tell me.
A	What did we do? We looked at the pictures and --
K	And we counted the pizzas and we counted the kids and --
H	We knew each one got one pizza.

K And we put a ring around.

H There, we're done.

K That was simple!

Pizza Problem Two

R This is pizza problem two. What I would like you to do is decide how much pizza each child will get if they share the pizza fairly. Notice these are the boys and their pizza and over here are the girls and their pizza.

K This is harder.

A No. This is going to be easy. Seven - eight - nine.

R What's your first question?

All (Read all the questions out loud at various speeds: (1) to self, (2) to others.)

K Well, you'd better do the boys first.

A Okay.

K Three. No.

A & K One piece each.

All Three.

A And I know how to make it too.

K I don't.

A You go like that, like that, like that hmmmm. (Partitions in thirds.) There. Like that. I made it.

R You'd better show the others how to do it.

H How did you do it?

A There. (Shows her work.)

H How did you do it?

K Just look it. (She shows her work.)

A Those aren't even. Tricky, but not for me because I did it before. I did it when we were playing on the board.

K I did it! (She shows her work.)

A It's still not even Karla.

K Look, one - two - three. There's three there.

A Yes, but they're not fair.

H Let's see.

A There's three, but they're not fair shares.

H Let's see yours.

A Here's the pizza and there's the middle and so you go (she begins to divide the pizza into three fair shares) like that, like that and that.

K Oh, I see.

H Oh, I see.

K (Passes her work over.) Is that better Annie?

A (Looks at the pizza.) No.

H I know another way of making three fair shares. See, like this. You can make a "T" like this. One - two - three.

A There's three but they're not fair shares.

K Let's see yours.

A Okay, here's the pizza and here's the middle. And you go there, there, there. (Demonstrates)

K Oh, I see. (Erases and tries again.) There, is that better Annie?

A No!

H I know another way of making three. See? Like this. You can make a "T" like this. See? One - two - three.

A A "T"? Okay, now here's the circle. (Demonstrates again.)

H One - two - three - four. Hey, that's not fair! That's not fair! (Looking at her own work.)

A Because they're not fair, they'll fight.

R Well, you've got the boys figured out, so now get the girls figured. (They read the questions aloud again, continually, to themselves and to the others.)

H There's 1, 2, 3, 4, 5, 6, 7, 8, 9 girls.

A There are nine girls.

H Ten girls! And there's only three pizzas.

K Three girls will share one pizza and three girls will share one pizza and the other three will share the other pizza.

H Yeah.

A (Monitors, recounting the girls and counts nine.) Nine - there's nine kids.

K Three plus three plus three equals nine.

H So these guys can share three and these guys can share three and --

A Okay, so there's one, two, three; one, two, three; one, two three. Oh, oh, I didn't make it right this time. Where's the eraser? Here it is. (Erases.) Now let me check how I did it last time. There's one going down like that to the middle and one going like that and one going like that.

K Like that Annie?

A (Looks) Nope.

K Oh. (Erases)

H I did my own.

K There, I did it. I did it! No good. Annie, how did you do that? Will you do it for me?

A No way.

R Why not? Why not show them how to do it? Show them carefully how to do it.

H That's sharing and that's co-operation.

A You make a circle here, and here's the circle. Look at the circle, there. Now there's the pizza and here's the middle and then take the middle of the pizza and go like that and that one goes like that and that one goes like that.

K Like this, Annie?

A Yup, that's right.

K That's right?

A Yup.

H (Talking to herself.) No, there's four. Can I use the eraser, Angie? Okay Karla, now there, there's the middle --

K I KNOW how to do it!

H (Working on her own.) If each child (reads the question to herself) --

H Three each.

K I did it. Look!

A No. Look! Still -- look. See?

K There - there - there. (She's re-drawing the pizza into thirds.)

R Let's go back. How much pizza does each boy get?

K One piece each.

R Out of how many?

K One pizza.

A One pizza.

R One piece out of one whole pizza. That's right. But also, one piece out of how many pieces?

A Oh.

H Hmmmm.

K One piece out of --

A (Counting the pieces.) One, two, three.

All One out of three.

R Is there a name for that?

H Wait a second, I'm not done. I'm not finished - I've got to record this. (To herself more than anyone.)

K (Still drawing thirds.) Like that Annie?

A Hmmm.

K There, like this? I did it Annie. See Annie? See, look, one - two - three; one - two - three.

A I'll show you one more sample. Now watch. See? (Shows her.)

K Okay, show me.

A There's the middle and you go out and then like this and then straight down.

K (Self talk. Inaudible.) (Shows her drawing.) Is that fair?

A No, you did it wrong. (Corrects the drawing.) Like that - there.

K Oh, I know now. Hmm, hmmm, hmm. (Monitors her drawing.) You go up and then you go -- like this Annie?

R How much pizza did each boy get?

K One piece each.

R Out of?

All One pizza.

R How many pieces in one pizza?

A Three.

K Three.

H Three.

R So that's one out of?

All Three.

K (Still working on drawing thirds.) I'm worried about this drawing. Hmmm. I don't draw very well. (Erases) (Self talk) That goes up, that goes out and that goes like that. Like that, Annie?

A Yup.

H (Busy spelling "piece.") One piece P - I - E - C - E each. Okay, I'll tell you what I wrote. Okay, how much pizza does each boy get? I'll tell ya. I wrote - "They each get one piece out of three pizzas." And then for girls I wrote - "Each girl gets one piece out of three pizzas."

R Out of three pizzas?

K Out of one pizza. There's three pizzas.

H Three pizzas! There's three pizzas!

K Out of one pizza.

H There's three pizzas and each girl gets one from out of three pizzas.

A One pizza.

K Each girl gets one out of one pizza. Each girl gets one out of one whole pizza.

H BUT there's three pizzas.

K One group of three gets one, one group of three gets one and one group of three gets the other one.

H Yeah, each girl gets one piece out of one pizza.

R And the boys?

A They each get one piece out of one pizza.

H One boy gets each one piece of pizza?

R Out of how much?

H One.

K One pizza.

R Do your next question.

(They read the next question, Does each child get a fair share? They're all whispering and talking to themselves.)

K They get the same amount.

A Karla, read the question again, very loud and to everyone.

K (Karla reads.) Each person would get the same amount.

R How would you know?

K Because there are three pizzas for the girls. One group of girls would get one - one group of three girls would get one pizza, one group of three would get the other and the other group of three would get the other.

R What about the boys?

K The boys -- well, we split it in three pieces so one boy can have one slice, the other boy can have the other slice and the other boy can have the other slice.

R Do you all agree? Well, how did you figure it out?

K Because we counted the pizzas and we counted the things together.

H And we counted them all and see, they all get the same. The girls get as much pizza as the others.

A Wait a minute, wait a minute. If each child gets a fair share, what about this?

K Does each boy get as much pizzas as each girl. How did you figure it out?

A Yes. And then how did you figure this out?

K No, we're not supposed to do that.

Pizza Problem Three

R This is pizza problem three. What I would like you to do is decide how much pizza each child will get if you share the pizza fairly. Notice these are the boys and their pizza and over here are the girls and their pizza.

(They all read the question out loud, seemingly to themselves and then to each other.)

K Hmmmm. Cause since there's two pizzas and we have three boys --

H They each get three. They each get three pieces. I'll show you. See?

A No.

H This guy gets one - two - three.

A No, not necessarily. Oh! Each boy gets two pieces.

H Yeah, that's what I meant. See? One, two. One, two for him, one, two for him, one, two for him.

K Do you agree on that?

All Yeah.

H T - W - O.

K Put the number.

All There are 1, 2, 3, 4. (Each counts aloud, singlely to 12. Helen is slower.) 7, 8, 9 --

K How do you spell pieces?

A P - I - E. So there's two pizzas for each group of girls.

K How much pizza does each girl get? (Counts to herself.) One, two, three; one, two, three. Oopsee. One, two, three. Everybody just stop and listen for a minute. It says how much pizza does EACH girl get? We're supposed to count the girls and then count the amount of pizzas and then if there are eight girls, say, okay, so --

H 1, 2, 3, 4, 5, 6, 7, 8, 9 girls.

K Twelve girls.

A There is?

All (Count aloud.)

K Twelve girls and there are only 1, 2, 3, 4, 5, 6, 7, 8 -- and there are eight pizzas. Four plus four equals eight - each girl would get two pieces! Each girl would get one piece each? Who agrees on that?

H Not me.

A We haven't figured it out yet - how to do it.

K Okay, there are eight pizzas and twelve girls. And four and four is eight. Four plus four is eight. You guys, four and four is eight. There are 1, 2, 3, 4 -- 8. This is a hardy!

H This girl can have one and this girl can have one and this girl can have one --

K This girl has one, this girl can have one, this girl can have one --

H See what I mean, Kara?

A Whoops, that's not right.

K Four plus four is eight. Four plus four is eight, so 1, 2, 3, 4, 5, 6, 7, 8. There aren't enough pizzas. So that means we have to take away.

A I'm making them into groups. There are two pizzas for each group of three kids.

K Who agrees on that?

H Me.

K Me. What did you say Annie? (Begins to monitor writing "There are.") Should we write 1, 2, 3, 4, there's 12 girls?

H There will be 12 pieces left. (Counts aloud.) 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13 -- oh.

A Yeah, that's right. I counted it.

K Each girl gets one piece.

H No, each girl gets two pieces. Yeah. We figured it out.

K Yeah, each girl gets two pieces. Who agrees? Do you agree that each girl gets two pieces?

H Yeah.

A Yeah.

K (Monitors as she writes the answer.) Each girl gets two pieces. P - I - E - C - E - S.

K (Reads question aloud.) If each child gets a fair share of the pizza, who would get more pizza? Each boy or each girl? Each of them get the same.

H Each of them get the same.

R Now how did you figure that out. Annie, how did you figure it out? What's the first thing you did?

A I made them into boxes.

H We made them into boxes and then we figured it out - then we figured out how much each girl got.

K You had to put - 'T' in the centre.

A I put the girls into boxes and the pizzas into boxes of two and then I cut the pizzas into -- Then I put the pizzas into pieces.

K She divided them in three. Two pieces each.

A Yeah.

R And how did you figure it out Karla?

K Like, well, I counted them out and there are three pieces in each one, right? Each one here and you divide it --

H You never drew it.

K I did it in my head! There are two here. There are only two pizzas and two -- well two pizzas and three girls so I divided them into pieces - three - pieces I had three pieces in each one. And then there are two in there and two in there and there wasn't enough left. So in one I had one more left and in the other I had one more left. So then I took them and I added them together and that made two so each girl got two. One got two and this one got two and this one got one from there and one from there - that's two!

R Are you tired or would you like to do another one?

All We want to do another one.

Pizza Problem Four

R This is pizza problem number four. Tell me how you would share that pizza.

H There's more boys than girls this time. (All read the question aloud.)

K This is what I do. I don't want to read all of them at the same time so I go like this. How much pizza does each boy get? (She covers up the others with paper.) And then I move it down and I read the next one. Be quiet - the thing is on. How much pizza does each boy get? Now the first one says "How much pizza does each boy get?" (Repeats)

A Hmmm.

K Now the first one says how much pizza does each boy get?

H Well, there's 1, 2, 3, 4.

A Four pizzas and I made a dot in the middle okay. So this one would belong to these guys and this one would belong to these --

K Yeah.

A These two would belong to these three.

K Yeah.

H Well, there's three here and there's three here and there's one guy left, Karla, and he eats a lot.

K (Counts aloud.) One, two, three; one, two, three; one oh, oh, we've got a problem.

H This is a problem with no answer.

K Oh, this problem's hard. Seven boys.

A Yeah, seven boys. I don't know how many pieces there are. I have to erase it.

H No, no, no, no. Two for them, two for them, two for them. Each boy gets two. This boy gets a pizza and he will share with this guy, and this boy gets a pizza and he will share with this guy and this boy gets a pizza and he will share with this guy. This boy gets a pizza and he doesn't have anybody to share with so what he will do is --

R So how will you share them so that they are even?

A One, two, three. One, two, three, four. That makes seven. (Tries thirds and quarters.)

H Yeah, for sure. Yeah, that's fair.

A If that's into four pieces?

H One, two, three.

All (Count the number of pieces aloud.)

H Yup. Well, there's three pieces here and four pieces here. One's left and mum can put that in the freezer.

K There's six men here - I've got a better idea. I have an idea. Okay. Since there are seven boys and there are only four pizzas, two boys can go together. Two boys can go together and this one --

H (Interrupts.) And then there will be three left over.

K Then these two boys can go together and these two can go together and this one can be left over, and then -- I've got it Angie!

A What are you going to do if there's pizza left over?

H Well, these two can go together.

K These two can go together.

R What did you say Angie?

A Two pieces for each boy.

H How did you do that?

A And then they get two and they get two and they get two and there's one there. And there's one piece of pizza left over they can give to the dog. (Still working.) And there's one there and one there.

K Do you agree Annie? Okay, we divided the pizzas in half. See there's three boys here - this boy can have one, this boy can have one. This boy can have one and then there's some left so this boy can have one. This boy can have one and that boy can have that. And that boy can have that.

H I have a better idea.

R (To Karla.) Is that fair?

A I don't know. I'm not done yet.

H Yup, mine is fair.

K Mine is fair. (Checking the pieces by counting aloud again.) (To Helen.) How come you're making X's.

H 1, 2, 3, 4; 1, 2, 3, 4; 1, 2, 3, 4; 1, 2, 3, 4.

K All these four boys can share out of these two and these three boys can share out of these two.

A I think we should take these pizzas and make them into three. We can go --

H Four pieces in this one, four pieces in this one --

K I'll read the question.

H We'll have two pizzas with four and two pizzas with three. Come here, I've got the answer. These guys can share that four, these guys can share that four and these guys can share this four. That solves it.

R Maybe you'd better take a look at the girls.

A Oh, this is simple.

K Yeah, we did this in the other one.

H You just cut the two pizzas in half and these guys can have two and these guys can have two.

R So why don't you do the same thing for the boys? (Blank stares.) Have you tried?

All Yeah.

K We did two. These guys had two, these guys had two. These guys had two. And there's one and one left over. Who agrees on each girl gets one-half each? Who agrees on that?

H (Talking about the boys.) These two boys can cut the pizza in half. These two boys can cut the pizza in half. These two boys can cut the pizza in three halves. I put three pizzas into four and one pizza into three.

K There will be one more piece left.

H I had an answer but we cut this one in half and this one in half and this one - and then there will be four pieces left.

K I have an idea. This is what we can do. Since there are four people, we pick four people and we give them all half. And then there will be one left and then all we have to do is take that one and put it in three pieces. And then there will be two more pieces and one person can have one half and one person can have the other.

R So tell me how much each boy gets.

K Each boy gets one pizza and another boy gets one half of a pizza and another boy gets --

A Wait, wait a minute. That doesn't sound fair to me. One boy gets a pizza and the other boy only gets half a pizza. That does not sound fair.

R So take a look at the girls. How much pizza does each girl get?

K For the girls I have an answer. Each girl gets a half pizza. Each girl gets a half a pizza - a half of one pizza. Do you agree?

A There are four girls and there are two pizzas and we can split one pizza in half and then the other and then that would be fair. I split the pizzas into four and I had two groups and each girl gets two pieces.

K But that's the same as this, Annie! When you split it in half it's still the same thing - you get two pieces and that's the same as one half.

R So how much does each girl get?

K One.

R One whole pizza?

K One slice each. Each girl gets a half a pizza. Each boy gets a half of a pizza.

R A half?

A A half I think. I'm not the one who figured it out. Karla figured it out.

R So who gets more?

H Each girl.

K Each boy.

A Karla, we haven't figured out the boys yet.

H Let's skip the boys out.

A No way.

H Okay, let's just read the next question.

K We can't do that Helen, until we do the boys.

H (Reads "How did you figure it out? Answer below.")

K Helen, that's what we are trying to find out.

H I think we should split this into four.

K What I agree is two boys get a half of this pizza, two boys get a half of this pizza and these three boys get --

H Hmmm.

K Two pizzas.

R You're awfully close.

K Since there are three boys left --

A Yeah, hey, yeah.

K We put a half to this boy and a half to this boy. And then we get this pizza and this boy can eat just one-half and one-half will be left over.

R So, how much does each boy get?

All One-half of a pizza.

R So what are you going to do with the half that's left over.

A I know! They can share it!

K How?

H Split it in half.

A Split it in more pieces.

H No! Each take a bite out of it.

R Okay, so who gets more?

K The girls.

A The boys.

R Which one, the boys or the girls?

A The boys, because each boy gets a half.

H Yeah, each boy gets a half of the pizza and after they eat the half then they can each get a little bite of the other half. After they each have one half a pizza then they each get a little bite of the other half so the boys get it. The boys get more.

APPENDIX F

PROTOCOL: GRADE FOUR, GROUP

GRADE FOUR

GROUP: MATHEW, WILLIAM, BRENDA

Pizza Problem One

M A pizza each.

B & W A pizza each.

All (Read the questions aloud and answer in sequential order.)
- both - neither - the same amount.
- Yes.
- How did you figure it out?

W Two pizzas and two boys and then each boy gets a whole pizza and then there's five more pizzas and there's five more girls so they don't have to divide it cause you can see that they each get a whole pizza. Instead of having to divide it into pieces -- they each get a whole pizza.

M Yeah!

B Like if there was one pizza you would have to divide it into five pieces - maybe six and one left over but here they each get one whole pizza.

Pizza Problem Two

All Divide it into three.
Divide it into three.

B Okay. How much pizza does each boy get?

All One third.

M Write down one third.

W I'm drawing mine. Where's the eraser? It's not even. There.

R How much pizza does each girl get?

All One third.

M A third each.

B (Reads aloud) "If each boy --"

W & M Neither.

B The same amount.

R Is there a word for the same amount?

All Equal.

B (Reads aloud) "Does each boy get as much pizza as each girl?"

All Yeah.

M (Reads aloud) "How did you figure this out?"

B We divided it into one-thirds. Like the boys got one-third because they wouldn't all get the same amount if you cut it into four pieces. Then there would be one piece left over. And if a piece is left over, they'd probably be arguing.

M Who'd get the last piece?

W There's three pizzas and twelve girls.

M No, nine girls.

W And if the boys got one-third of a pizza each and there's three pizzas for the girls then divide the girls into thirds and each third of the girls gets one pizza. Then you divide that pizza into the thirds like the boys.

Pizza Problem Three

B (Reads) "How much pizza does each boy get?"

W (Moans)

M I think I have a solution for this. Sounds a little wacky but -- Since there's only three boys and two pizzas these boys could get two-thirds. Like you can make it into thirds and then this boy gets one-third from this pizza and one-third from that.

W Right. (Whispering. Erasing.) (Monitoring own work.)

B Each boy would get two pieces of pizza.

M Each piece is one-third but they each get two-thirds in all.

W	Now, what about the girls? (Counting girls and pizzas.)
M	It's the same as the boys.
All	(Drawing)
M	Okay. That's two-thirds.
B	This piece. This piece. And this piece for the third girl.
W	Three girls and two pizzas.
All	(Self talk: monitoring.)
W	Okay there's 12 girls.
B	You can see.
W	Three girls to two pizzas.
M	So the same as the boys.
W	Each girl gets two-thirds of a pizza.
B	Yup.
All	(Read and answer each question in sequential order.) How did you figure this out?
B	Figured it out to split it up into three thirds and then if one boy got one piece of pizza, then the other boy got one and the other boy got one.
W	There would be three pieces left over.
B	Yeah.
W	There's twelve girls. An even number. And an even number of pizzas.
R	What made you decide to try thirds on the girls?
M	Well, because there's sets of three on the girls and sets of two on the pizza.

Pizza Problem Four

W Oh. Look!

M Gessh! (Pencil tapping, while monitoring detail.)

W Do the girls first.

M Yeah. They're the easy ones.

W Each girl gets one-half of a pizza.

B Yeah. Mathew, right. Now with the boys -- (Monitor the detail.)

W Wait that might be it. Won't work the same.

M I think this will work. No it wouldn't -- could everybody get the same amount? Okay. (Tries solution of girls.) That wouldn't add up the same.

W We haven't tried fifths yet.

M I have an idea.

B (Monitors detail, taps.)

M Put it into sevenths. (Taps)

B Seven boys. (Taps)

M Put it into sevenths -- so it's seven pieces.

W Uh, uh. You can't divide an uneven number (seven) into an even number (four).

M Why not? You can divide it. You could do that. Then they each get four pieces.

B Mathew, there's seven boys and there's four pizzas.

M I know. Divide each one into seven and then they each get four pieces.

W That won't work. There's an uneven number of boys and there has to be six boys.

M That's with numbers, not real things.

W Oh. Oh, yeah!

M Yeah, that would work. (Monitoring the activity.)

B Yeah, but what about the other pizzas?

M Do it the same way.

All (Monitoring.)

W Yeah, they each get four-sevenths of a pizza.

All (Read each and answer each in sequential order.)
1. Four-sevenths.
2. One-half.
3. Boys get more. Each boy.

B See four-sevenths is more than one-half.

W Yeah.

4. Each boy gets more. So -- no.

B 5. We counted out how many boys there was. Then we counted how many pizzas and there's four pizzas and seven boys as so we divided them into seven --

M And there's four pizzas and we did that on all four pizzas so each boy would get four pieces -- four-sevenths.

W And for the girls we split it in half.

B 'Cause there's two pizzas and four girls.

R How did you figure out who had more?

B Well four-sevenths is more than a half.

M 'Cause four is more than three and three takes up less than -- well the four must go into the half way mark of seven.

Pizza Problem Five

All (Monitor the detail.)

M Split it into six. Then they each get four -- they each get six pieces.

W It's almost just the other way around.

M There's four boys and there's six pizzas so you split that into -- fourths?

B (Monitors fourths.) It works.

M So they each get -- let's see --

W They each get half of a pizza.

B I know.

M They each get one and a fourth. A whole one and then a fourth.

B (Explains fourths by monitoring.) Split it into fourths then each one gets six.

W Each boy gets a whole pizza and a half.

B Is there another way?

W Six fourths.

M Six quarters.

All (Monitor the detail.)

W Nine divide by six is three. No!

M Remainder three.

B Yeah, nine divided by six is one, remainder three. Nine divided by six is one, remainder three.

W Do you already get division?

M Not in class but I learned it in grade 3.

W Wheew.

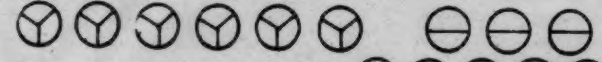
B (Monitors activity.) (Grouping girls.) I did it, I think. Okay, each girl gets four pieces of pizzas.

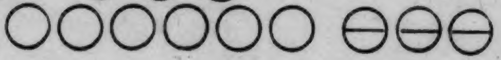
W Let's see.

B Because you give the girls each a pizza first and then the other three pizzas divide in halves.

W Two-thirds. Each person gets three thirds of a pizza. One whole.

M They get a pizza and a half. Instead of splitting these up in thirds, and then the remaining three you split in half. Give them a whole pizza. 'Cause there's only three left and there's six girls so you split them in half so there's six pieces. So each gets an extra piece.

B Look. 

M You have the same thing. See. 

B You can split these in thirds and split the other three pizzas in halves and then add and they get four pieces of pizza each. That's a whole and a half.

M One and a half.

W Yeah, they can have three thirds and a half or one pizza and a half. It's the same thing.

(Read and answer aloud each question in sequential order.)

M 1. 6/4 = six quarters = 1 1/2. It's all the same.
W 2. 1 1/2
B 3. Neither, get equal.
B 4. Neither gets more. Yes.
M 5. Did the boys first 'cause the girls were harder. Then we did the girls.

Pizza Problem Six

M This is easy.

W The girls -- thirds.

B Now the boys. Hmm.

All (Monitor the details.)

W Cut each pizza in seven.

B Give each boy three pieces.

M Three-sevenths, each boy gets three-sevenths.

(Read and answer aloud each question in sequential order.)

All 1. 3/7.
All 2. 1/3.
W 3. "Look the picture!" 3/7 is more. Well, a little. The boys get more.
M 4. Easy. Thirds for the girls. Then cut in seven for the boys.

Pizza Problem Seven

All (Monitor the detail.)

M Do the boys first. Give each boy one pizza. Cut the other two in thirds. Then give each boy two pieces. Each boy gets one pizza and two-thirds.

"Now the girls."

All (Monitor the detail.)

W Give each girl one pizza. Cut the remainder in five, fifths. Give each girl three pieces.

B Each girl gets one pizza and three pieces.

M One pizza and three-fifths.

(Read and answer each question aloud in sequential order.)

M & W 1. 1 and 2/3. Yeah 5/3 too!

All 2. 1 and 3/5.

B 3. It's real close.

W I think two-thirds.

M Look. Yeah, 1 2/3 is more.

All The boys get more

W 4. The boys.

M & W 5. Give each boy -- cut up the remainder in thirds. Give them each two-thirds.

B Give each girl one and cut up the remainder in five; fifths. Give them three-fifths more.

Pizza Problem Eight

All (Monitor the detail.)

M Give each boy one and cut the other one in four.

B Yeah! 'Cause there's four boys.

W Now the girls.

B Give each girl one and cut the left over in five. Humph.

M They each get 1 1/5.

W They do?

M Yup. See.

W Right.

(Read and answer each question in sequential order.)

All 1. One and one fourth.

All 2. One and one fifth.

All 3. The boys. No.

All 4. The boys.

M 5. Give each boy one and cut the remainder in four.
Give each girl one and cut the remainder in five.

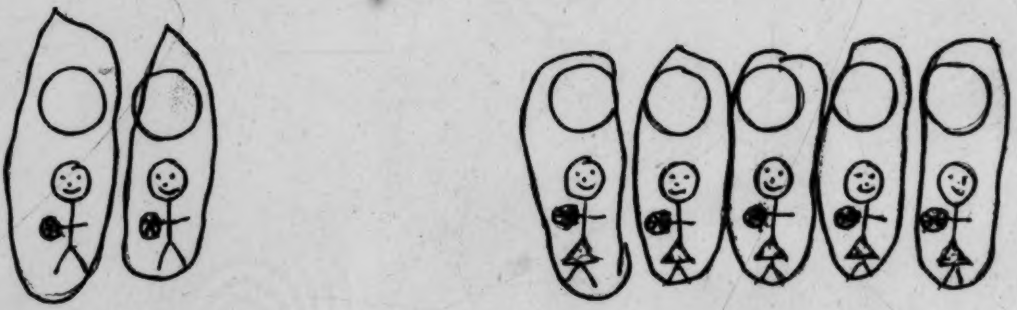
APPENDIX G

SAMPLE OF STUDENT SOLUTIONS

Grade Two, Group

SHARING THE PIZZA

1.



How much pizza does each boy get? Each boy gets a hole pizza.

How much pizza does each girl get? Each girl gets a hole pizza.

If each child gets a fair share of the pizza, who would get more pizza - - - each boy or each girl? Same

Does each boy get as much pizza as each girl? Yes

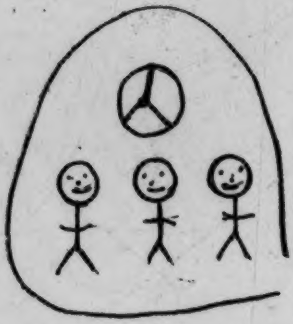
How did you figure this out?

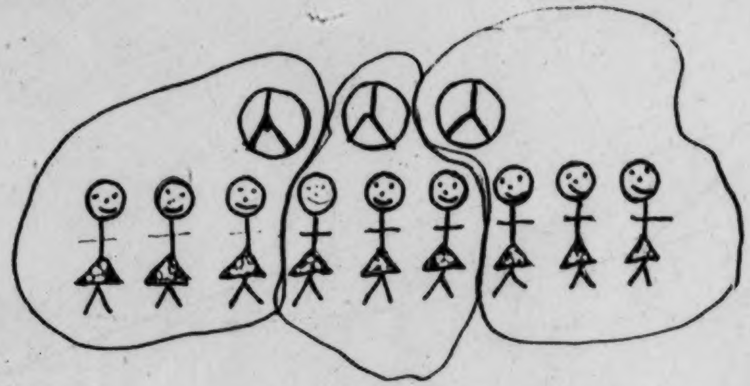
Answer below.

Grade Two, Group

SHARING THE PIZZA

2.





How much pizza does each boy get?

How much pizza does each girl get?

If each child gets a fair share of the pizza, who would get more pizza - - - each boy or each girl?

Does each boy get as much pizza as each girl?

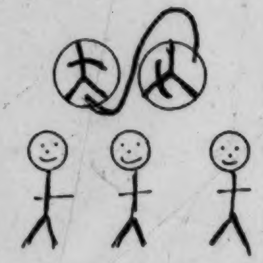
How did you figure this out?

Answer below.

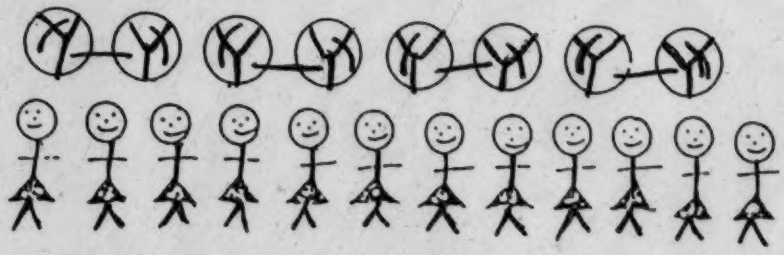
Grade Three, Group

SHARING THE PIZZA

3.



A



If each child gets a fair share of the pizza, who would get more pizza -- each boy or each girl?

Does each boy get as much pizza as each girl?

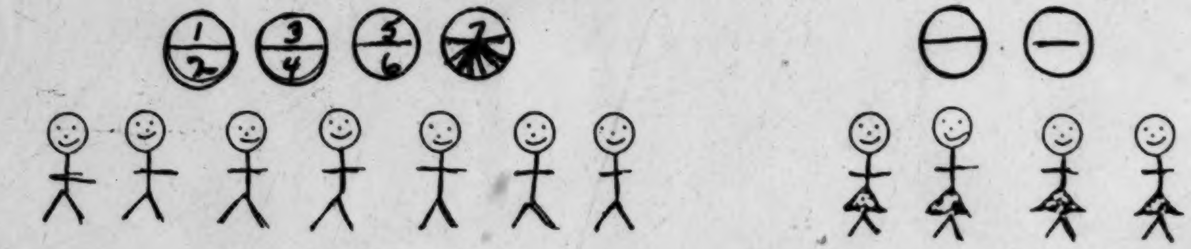
How did you figure this out?

Answer below

Grade Three, Group

SHARING THE PIZZA

4.



If each child gets a fair share of the pizza, who would get more pizza -- each boy or each girl? boy

Does each boy get as much pizza as each girl? more

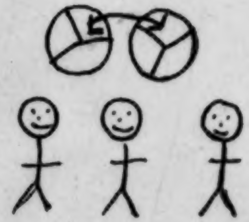
How did you figure this out?

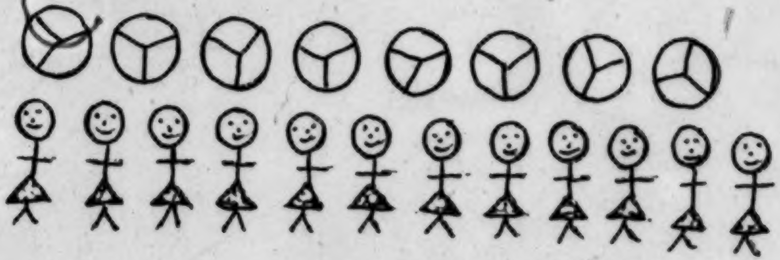
Answer below we divided a pizza in a half and a seventh

Grade Four, Group

SHARING THE PIZZA

3.





How much pizza does each boy get?

How much pizza does each girl get?

If each child gets a fair share of the pizza, who would get more pizza - - - each boy or each girl?

Does each boy get as much pizza as each girl?

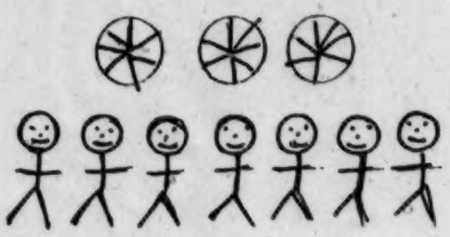
How did you figure this out?

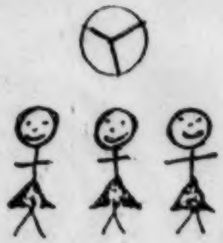
Answer below.

Grade Four, Group

SHARING THE PIZZA

6.





How much pizza does each boy get?

How much pizza does each girl get?

If each child gets a fair share of the pizza, who would get more pizza - - - each boy or each girl?

Does each boy get as much pizza as each girl?

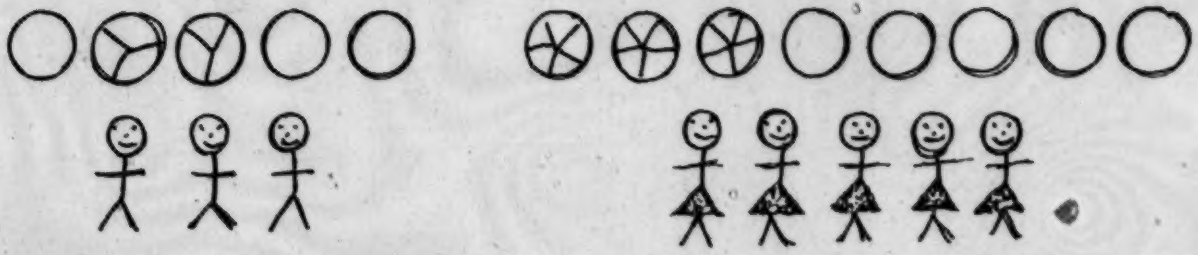
How did you figure this out?

Answer below.

Grade Four, Group

SHARING THE PIZZA

7.



How much pizza does each boy get?

How much pizza does each girl get?

If each child gets a fair share of the pizza, who would get more pizza - - - each boy or each girl?

Does each boy get as much pizza as each girl?

How did you figure this out?

Answer below.

APPENDIX H

FREQUENCY OF LANGUAGE FUNCTIONS

LANGUAGE FUNCTIONS USED TO UNDERSTAND THE MATHEMATICAL PROBLEMS

Table 1

Frequency of Language Functions Used by Grade Two, Three and Four Children Attempting to Understand the Mathematical Problems

		Type of Function					
		C^1		C^2		C^3	
Math Problem	Grade	I	G	I	G	I	G
1	2	1	3	1	6	0	3
2		2	7	1	8	0	2
3		0	4	0	6	2	8
4		0	5	0	0	1	3
5		0	*	0	*	0	*
6		0	*	0	*	1	*
1	3	0	4	0	4	1	1
2		0	0	0	0	0	1
3		0	0	0	0	0	4
4		0	0	0	0	2	2
5		0	*	0	*	2	*
6		0	*	0	*	1	*
1	4	0	0	0	0	5	0
2		0	0	0	0	0	0
3		0	0	0	0	1	2
4		0	0	0	0	0	3
5		*	0	*	0	*	2
6		*	0	*	0	*	2
7		*	0	*	0	*	1
8		*	0	*	0	*	1

I - indicates the individual child
G - indicates the group

C^1 - reading aloud to the self
C^2 - reading aloud to another
C^3 - examining the illustration

* - problem not presented

LANGUAGE FUNCTIONS USED TO PLAN THE SOLUTIONS TO THE MATHEMATICAL PROBLEMS

Table 2

Frequency of Language Functions Used by Grade Two, Three and Four Children Attempting to Plan the Solutions to the Mathematical Problems

		Type of Function					
		C^4		C^5		C^6	
Math Problem	Grade	I	G	I	G	I	G
1	2	1	4	0	0	0	1
2		1	4	0	0	1	16
3		0	17	9	2	2	14
4		3	20	5	7	1	14
5		6	*	2	*	0	*
6		5	*	2	*	1	*
1	3	1	2	0	0	0	3
2		0	2	1	3	0	4
3		4	5	4	2	2	8
4		5	5	4	3	1	3
5		4	*	1	*	3	*
6		3	*	2	*	5	*
1	4	0	0	0	0	0	0
2		0	0	5	0	0	2
3		0	3	3	0	0	0
4		0	9	2	4	0	9
5		*	8	*	8	*	6
6		*	3	*	1	*	3
7		*	5	*	0	*	3
8		*	7	*	0	*	3

I - indicates the individual child
G - indicates the group

C^4 - indicates use of language to report
C^5 - indicates use of language to predict
C^6 - indicates use of language to direct

* - problem not presented

LANGUAGE FUNCTIONS USED TO SOLVE THE MATHEMATICAL PROBLEMS

Table 3

Frequency of Language Functions Used by Grade Two, Three and Four Children While Solving the Mathematical Problems

		Type of Function					
		C^7		C^8		C^9	
Math Problem	Grade	I	G	I	G	I	G
1	2	2	3	4	4	0	1
2		0	16	0	43	0	17
3		0	0	0	1	0	1
4		1	6	1	5	1	0
5		1	*	1	*	0	*
6		1	*	2	*	1	*
1	3	0	0	0	1	0	0
2		1	0	7	1	1	0
3		0	3	2	3	0	0
4		3	1	1	2	0	0
5		1	*	2	*	0	*
6		2	*	5	*	2	*
1	4	0	0	2	0	0	0
2		2	3	1	1	1	2
3		0	10	1	3	0	0
4		1	3	0	1	0	1
5		*	8	*	13	*	0
6		*	0	*	3	*	0
7		*	0	*	3	*	0
8		*	0	*	1	*	0

I - indicates the individual child
G - indicates the group

C^7 - use oflanguage to direct
C^8 - use of language to report
C^9 - use of language to clarify

* - problem not presented

LANGUAGE FUNCTIONS USED TO REVIEW THE SOLUTIONS TO THE MATHEMATICAL PROBLEMS

Table 4

Frequency of Language Functions Used by Grade Two, Three and Four Children to Review the Solutions to the Mathematical Problems

		Type of Function	
		C^{10}	
Math Problem	Grade	I	G
1	2	5	18
2		16	32
3		16	19
4		8	28
5		10	*
6		11	*
1	3	5	14
2		4	9
3		18	23
4		9	14
5		13	*
6		6	*
1	4	12	10
2		31	15
3		5	6
4		17	17
5		*	7
6		*	7
7		*	14
8		*	9

I - indicates the individual child
G - indicates the group

C^{10} - clarifying ideas by questioning, reflecting and evaluating

* - problem not presented